日本数学会

2026年度秋季総合分科会

無限可積分系特別セッション
講演アブストラクト

（2日目／9月2日）

2026年9月

於　神戸大学

Riemann 球面上の線型微分方程式の分類

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1. スペクトル型

Fuchs 型方程式 $\mathcal{M}: \frac{du}{dx} = \sum_{j=1}^{p-1} \frac{A_j}{x-a_j} u$ は，その局所構造が一般化 Riemann scheme

$$\begin{Bmatrix} x = a_1 & \cdots & x = a_p = \infty \\ [\lambda_{1,\nu}]_{m_{1,\nu}} \ \ (1 \le \nu \le n_1) & \cdots & [\lambda_{p,\nu}]_{m_{p,\nu}} \ \ (1 \le \nu \le n_p) \end{Bmatrix}$$

で分かる．ここで，$A_j \in M(n, \mathbb{C})$, $A_p = -(A_1 + \cdots + A_{p-1})$ である．

なお，ordered すなわち $m_{j,1} \ge m_{j,2} \ge \cdots \ge m_{j,n_j}$ とすると

$$\operatorname{rank}(A_j - \lambda_{j,1}) \cdots (A_j - \lambda_{j,k}) = n - m_{j,1} - \cdots - m_{j,k} \quad (k = 1, \ldots, n_j)$$

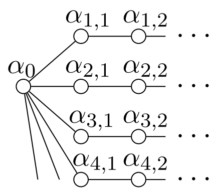
定義. 正整数 n の分割の p 個の組

$$\mathbf{m} : n = m_{j,1} + \cdots + m_{j,n_j} \quad (j = 1, \ldots, p)$$

を $\mathcal{M}$ の**スペクトル型**といい，$\mathbf{m} = m_{1,1} \cdots m_{1,n_1}, m_{2,1} \cdots n_{2,n_1}, \cdots m_{p,n_p}$ と記述する．

たとえば，一般化超幾何函数 ${}_3F_2(x)$ の満たす微分方程式のスペクトル型は $111, 21, 111$

2. 星型 Kac-Moody ルート系



$$(\alpha|\alpha) = 2 \quad (\alpha \in \Pi := \{\alpha_0,\, \alpha_{j,\nu} \mid j \ge 1,\, \nu \ge 1\})$$

$$(\alpha_0|\alpha_{j,\nu}) = -\delta_{\nu,1}$$

$$(\alpha_{i,\mu}|\alpha_{j,\nu}) = \begin{cases} 0 & (i \ne j \quad \text{or} \quad |\mu - \nu| > 1) \\ -1 & (i = j \quad \text{and} \quad |\mu - \nu| = 1) \end{cases}$$

$$s_\alpha \ : \ x \mapsto x - (x|\alpha)\alpha \quad (\alpha \in \Pi), \quad W := \langle s_\alpha | \alpha \in \Pi \rangle$$

$$\alpha_{\mathbf{m}} := n\alpha_0 + \sum_{j \ge 1} \sum_{i \ge 1} \Bigl(\sum_{\nu > i} m_{j,\nu} \Bigr) \alpha_{j,i} \qquad (\operatorname{ord} \mathbf{m} := n)$$

定理 [Crawley-Boevey]. $\mathbf{m}$：既約実現可能 $\Leftrightarrow$ $\alpha_{\mathbf{m}}$：正ルート（$\mathbf{m}$：分割不能 $\Leftarrow$ affine）

定理 [Katz]. $\dim \{(\tilde{A}_1, \ldots, \tilde{A}_p) \mid \tilde{A}_j \sim A_j,\ \sum_j \tilde{A}_j = 0\} / GL(n, \mathbb{C}) = 2 - \operatorname{idx} \mathbf{m}$

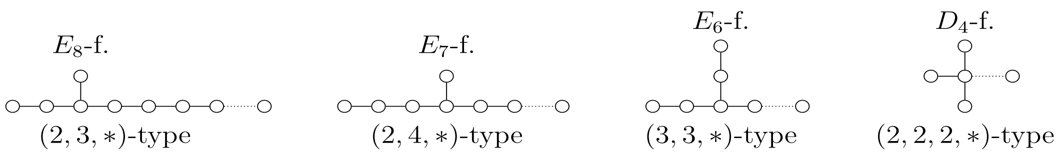
$\operatorname{idx} \mathbf{m} := (\alpha_{\mathbf{m}} | \alpha_{\mathbf{m}})$：rigid 指数　　($\operatorname{idx} \mathbf{m} = 2 \ \Leftrightarrow\ \mathbf{m}$：rigid)

定義. $\mathcal{L} := \{\mathbf{m}$: 既約実現可能で non-rigid な分割の組 $\}$

$\mathcal{L} \ni \mathbf{m}$：fundamental $\overset{\text{def.}}{\Leftrightarrow}$ $(\alpha_{\mathbf{m}}|\alpha) \le 0 \quad (\forall \alpha \in \Pi)$　($\Rightarrow \alpha_{\mathbf{m}}$: 正の虚ルート)

$\mathcal{L} \ni \mathbf{m}$：$(k_1, k_2, *)$-type $\overset{\text{def.}}{\Leftrightarrow}$ $p = 3$ かつ $(n_1, n_2) = (k_1, k_2)$

$\mathcal{F} := \{\mathbf{m} \mid \mathbf{m} : \text{fundamental}\}$, $\mathcal{F}^T := \{\mathbf{m} \in \mathcal{F} \mid \mathbf{m} : T\text{-fundamental}\}$　$T : E_8,\, E_7, \ldots$



定理 [O]. $\mathcal{F}_\ell := \{\mathbf{m} \in \mathcal{F} \mid \operatorname{idx} \mathbf{m} = -\ell\}$ $\Rightarrow$ $|\mathcal{F}_\ell| < \infty$　$(\ell = 0, 2, 4, \ldots)$

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3. 方程式の変換

不分岐不確定特異点を許す方程式のスペクトル型は「n の分割の組 + 細分構造」で表す

定理 [Hi]. 分割の組 + 細分構造：既約実現可能 $\Leftrightarrow$ 開折（分割の組）が既約実現可能

変換：middle convolution, addition, 合流/開折, Harnad dual，Möbius 変換を考える

対応するスペクトル型の変換（idx は不変）：

$$\mathrm{mc}_\sigma\,\mathbf{m} := \left\{m_{j,\nu} - d_\sigma(\mathbf{m})\delta_{\sigma_j,\nu}\right\}_{\substack{j=1,\dots,p\\ \nu=1,\dots,n_j}} \qquad \left(d_\sigma(\mathbf{m}) := \textstyle\sum_j m_{j,\sigma_j} - (p-2)\operatorname{ord}\mathbf{m}\right)$$

$$\mathbf{m} = m'_{1,1}\cdots m'_{n_1,R_{n_1}},\ m_{2,1}\cdots m_{2,n_2},\ m_{3,1}\cdots m_{3,n_3},\ \dots,\ m_{p,1}\cdots m_{p,n_p}$$

$$\underset{\text{開折}}{\overset{\text{合流}}{\rightleftarrows}}\ (\overbrace{m'_{1,1}\cdots m'_{1,R_1}}^{m_{2,1}})\cdots(\overbrace{m'_{n_1,1}\cdots m'_{n_1,R_{n_1}}}^{m_{2,n_2}}),\ m_{3,1}\cdots m_{3,n_3},\ \cdots,\ \cdots,\ m_{p,1}\cdots m_{p,n_p}$$

$$= m'_{1,1}\cdots m'_{1,R_1}m'_{2,1}\cdots m'_{n_1,R_{n_1}}|m_{2,1}\cdots m_{2,n_2},\ \underline{m_{3,1}}m_{3,2}\cdots m_{3,n_3},\ \dots,\ \underline{m_{p,1}}m_{p,2}\cdots m_{p,n_p}$$

$$\overset{\mathrm{Hd}}{\leftrightarrow}\ (m_{3,2}\cdots m_{3,n_3})\cdots(m_{p,2}\cdots m_{p,n_p}),\ \underline{m'_{1,0}}m'_{1,1}\cdots m'_{1,R_1},\ \dots,\ \underline{m'_{n_1,0}}m'_{n_1,1}\cdots m'_{n_1,R_{n_1}}$$

idx=0, 4 fundamental スペクトル型

6: 33,222,111111 $E_8^{(1)}$	4: 22,1111,1111 $E_7^{(1)}$	3: 111,111,111 $E_6^{(1)}$	2: 11,11,11,11 $D_4^{(1)}$

$33,222,111111 \xrightarrow{(1^3)(1^3)} 22,1111,1111 \xrightarrow{(1^2)(1^2)} 111,111,111 \xrightarrow{(1)(1)(1)} 11,11,11,11$

$(1^3)(1^3),2^3$ $(1^2)(1^2)(1^2),3^2$ $(1^2)(1^2),1^4$ $(1)(1)(1)(1),2^2$ $(1)(1)(1),1^3$ $(1)(1),11,11$

$((1)(1))((1)(1))$ $((1))((1))((1))$ $(1)(1),(1)(1)$ $((1))((1)),11$

$(((1)))(((1)))$

$$33,222,111111 \xrightarrow{\text{合流}} 111111|33,222 = (111)(111),\underline{2}22 \xrightarrow{\mathrm{Hd}} (22),\underline{1}111,\underline{1}111 \xrightarrow{\text{開折}} 22,1111,1111$$

定義. $\mathbf{m}, \mathbf{m}' \in \mathcal{L}$ に対し，$\mathbf{m}\sim\mathbf{m}' \overset{\text{def.}}{\Leftrightarrow}$ 上の変換の組み合わせで $\mathbf{m}$ と $\mathbf{m}'$ は移り合う

問題. $\mathbb{P}^1$ 上の線型微分方程式（ただし不確定特異点は不分岐とする）の全体を，これらの変換で移り合うものに分類せよ．すなわち $\mathcal{L}/\sim$ を決定せよ (cf. $\mathcal{L}/\underset{\mathrm{mc}}{\sim} \simeq \mathcal{F}$, [Ka])．

idx=−2, 13 fundamental スペクトル型 ⇒ 2 classes

8: 44,332,11111111 $[11]_5^T$	6: 33,2211,111111	5: 32,11111,11111	5: 221,221,11111
4: 211,1111,1111	4: 31,22,22,1111 Sa	3: 21,21,111,111 FS	2: 11,11,11,11,11 $[11]_5$
12: 66,444,2222211 $E_8^{(2)}$	10: 55,3331,22222	8: 44,2222,22211 $E_7^{(2)}$	6: 222,222,2211 $E_6^{(2)}$
4: 22,22,22,211 $D_4^{(2)}$			

4. 結果

定理 i) $\mathcal{L} = \bigsqcup_{\mathbf{n}\in\mathcal{F}^{E_8}}\{\mathbf{m}\in\mathcal{L}\mid\mathbf{m}\sim\mathbf{n}\}$ （すなわち $\mathcal{L}/\sim\ \simeq\ \mathcal{F}/\sim\ \simeq \mathcal{F}^{E_8}$）

ii) $\mathbf{m}\in\mathcal{F},\ \mathbf{n}\in\mathcal{F}^{E_8},\ \mathbf{m}\sim\mathbf{n},\ \mathbf{m}\neq\mathbf{n}\ \Rightarrow\ \operatorname{ord}\mathbf{m} < \operatorname{ord}\mathbf{n}$.

iii) $\mathbf{m}\in\mathcal{F}^T$ $(T = D_4,\ E_6,\ E_7)$ に対し，$\mathbf{m}\sim\mathbf{n}$ となる $\mathbf{n}\in\mathcal{F}^{E_8}$ の具体型が分かる．

たとえば，$\mathbf{n} = \bar{m}_{11}\bar{m}_{12},\ \bar{m}_{21}\bar{m}_{22}\bar{m}_{23},\ \bar{m}_{31}\cdots\bar{m}_{3n_3}\in\mathcal{F}^{E_8}$ に対し，$\mathbf{n}\sim\exists\mathbf{m}\in\mathcal{F}^{D_4}$

$\Leftrightarrow\ \bar{m}_{11} = \bar{m}_{12},\ \bar{m}_{21} = \bar{m}_{22},\ \bar{m}_{23} = 2\bar{m}_{31} = 2\bar{m}_{32}$ （このとき $\mathbf{m}$ は一意に定まる）

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箙の合流に由来する 4 次元 q-ガルニエ系の退化

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概　要

q-ガルニエ系はまず坂井によって提唱され, 続いて長尾・山田によって離散時間発展の方向を増やすように拡張された. その後, これらの時間発展がアフィン・ワイル群の増田・大久保・津田による双有理表現に由来することが, 大久保と講演者によって明らかにされた. 本講演では, 4 次元 q-ガルニエ系の退化の幾つかを, その由来となる箙の頂点の合流を用いて定式化する.

1. 4 次元 q-ガルニエ系

図 1 の箙 Q_{12} において, y_i を頂点 i に対応する係数, μ_i を頂点 i における変異, (i,j) を頂点 i,j の互換, ι を全ての矢の反転とする. 定義の詳細については [3] を参照のこと. これらの変換を用いて, [2] に従って, 単純鏡映とディンキン図形の自己同型を

$$
\begin{aligned}
&r_i = \mu_{2i+1}(2i+1, 2i+2)\mu_{2i+1} \quad (i = 0, \ldots, 5),\\
&s_0 = \mu_1\mu_4\mu_5\mu_8\mu_9(9,12)\mu_9\mu_8\mu_5\mu_4\mu_1, \quad s_1 = \mu_2\mu_3\mu_6\mu_7\mu_{10}(10,11)\mu_{10}\mu_7\mu_6\mu_3\mu_2,\\
&s'_0 = \mu_1\mu_3\mu_5\mu_7\mu_9(9,11)\mu_9\mu_7\mu_5\mu_3\mu_1, \quad s'_1 = \mu_2\mu_4\mu_6\mu_8\mu_{10}(10,12)\mu_{10}\mu_8\mu_6\mu_4\mu_2,\\
&\pi_1 = (2,4,6,8,10,12)(1,3,5,7,9,11), \quad \pi_2 = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12),\\
&\pi_3 = \iota(1,12)(2,11)(3,10)(4,9)(5,8)(6,7)
\end{aligned}
$$

と定める.

事実 1 ([2]). 群 $\langle r_0,\ldots,r_5\rangle$, $\langle s_0,s_1\rangle$, $\langle s'_0,s'_1\rangle$ はそれぞれ $A_5^{(1)}$, $A_1^{(1)}$, $A_1^{(1)}$ 型アフィン・ワイル群と同型である. 更に, どの 2 つの群も互いに可換である.

更に, 平行移動を

$$
\begin{aligned}
&T_i = r_i r_{i+1} r_{i+2} r_{i+3} r_{i+4} r_{i+5} r_{i+4} r_{i+3} r_{i+2} r_{i+1} \quad (i = 0, \ldots, 5),\\
&U_0 = s_0 s_1, \quad U_1 = s_1 s_0, \quad U'_0 = s'_0 s'_1, \quad U'_1 = s'_1 s'_0, \quad V = \pi_1 r_5 \ldots r_1 s_1, \quad V' = \pi_2\pi_1 r_5 \ldots r_1 s'_1,
\end{aligned}
$$

ただし, $r_{i+6} = r_i$ と定める. 平行移動は拡大アフィン・ワイル群の極大可換部分群を生成し, その双有理表現は非線形 q-差分方程式系の族を与える. 本講演では, この族を 4 次元 q-ガルニエ系と呼ぶ.

2. 箙の合流と q-ガルニエ系の退化

2 次元の q-パンルヴェ方程式の退化が箙の頂点の合流に由来することは, [1] において初めて指摘され, その後 [4] において具体的な計算が与えられた. これらの先行研究に従って, 2 つの頂点 i,j の合流 $i \to j$ を次のように定める.

1. i と j の間の辺を全て消去する.
2. 更に, i と j を重ねて新たな 1 つの頂点 j とする.

本研究は科研費 (課題番号:25KJ0371) の助成を受けたものである.
2010 Mathematics Subject Classification: 39A13, 13F60, 17B80, 34M56
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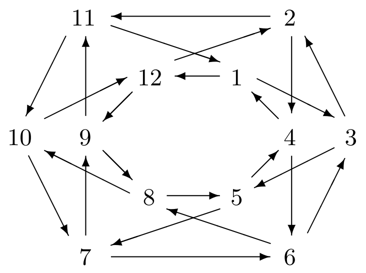

図 1: 箙 Q_{12}

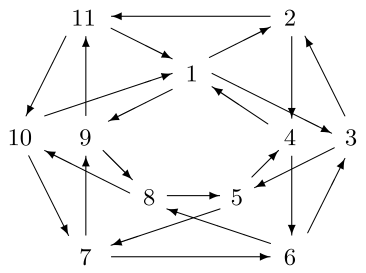

図 2: 箙 Q_{11}

合流 $i \to j$ は, 係数に対しては次のような操作と定める.

1. $y_i \to \varepsilon^{-1} y_j$, $y_j \to \varepsilon$ と置き換える.
2. 更に, 極限 $\varepsilon \to 0$ を取る.

箙 Q_{12} において合流 $12 \to 1$ を行うと, 図 2 の箙 Q_{11} が得られる. なお, 箙 Q_{12} における合流は箙の形の対称性から $12 \to 1$ と $12 \to 2$ の実質 2 つだけしかなく, 後者も変異によって Q_{11} に帰着する. 箙 Q_{11} において, [2] に従って, 単純鏡映とディンキン図形の自己同型を

$$r_0 = \mu_1\mu_2(2,11)\mu_2\mu_1, \quad r_i = \mu_{2i+1}(2i+1, 2i+2)\mu_{2i+1} \quad (i = 1, \ldots, 4),$$
$$s_0 = \mu_1\mu_4\mu_5\mu_8(8,9)\mu_8\mu_5\mu_4\mu_1, \quad s_1 = \mu_2\mu_3\mu_6\mu_7\mu_{10}(10,11)\mu_{10}\mu_7\mu_6\mu_3\mu_2,$$
$$\pi_1 = (1,3,5,7,9,11,2,4,6,8,10)\mu_2, \quad \pi_2 = \iota(2,11)(3,10)(4,9)(5,8)(6,7)$$

と定める.

事実 2 ([2]). 群 $\langle r_0, \ldots, r_4 \rangle$, $\langle s_0, s_1 \rangle$ はそれぞれ $A_4^{(1)}$, $A_1^{(1)}$ 型アフィン・ワイル群と同型である. 更に, これらの群は互いに可換である.

更に, 平行移動を

$$T_i = r_i r_{i+1} r_{i+2} r_{i+3} r_{i+4} r_{i+3} r_{i+2} r_{i+1} \quad (i = 0, \ldots, 4),$$
$$U_0 = s_0 s_1, \quad U_1 = s_1 s_0, \quad V = \pi_1 r_4 r_3 r_2 r_1 s_1, \quad V' = \pi_1^5 s_1,$$

ただし, $r_{i+5} = r_i$ と定める.

定理 3 ([3]). 箙 Q_{12} における合流 $12 \to 1$ によって, 単純鏡映とディンキン図形の自己同型は次の表のように退化する.

Q_{12}	$r_0r_5r_0$	r_1	r_2	r_3	r_4	s_0	s_1	$\pi_2 s_0'$	π_3
Q_{11}	r_0	r_1	r_2	r_3	r_4	s_0	s_1	π_1^5	π_2

また, 平行移動は次の表のように退化する.

Q_{12}	T_5T_0	T_1	T_2	T_3	T_4	U_0	U_1	V	$(V')^{-1}V$
Q_{11}	T_0	T_1	T_2	T_3	T_4	U_0	U_1	V	V'

本講演では更に, 箙 Q_{11} における合流を考える. これにより, 10 個の頂点を持つ箙が全部で 5 つ得られる. また時間が許せば, q-超幾何級数による特殊解の退化についても言及する.

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離散パンルヴェ方程式の退化図式の多変数化

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楕円差分パンルヴェ方程式の多変数版として，楕円差分ガルニエ系と呼ばれる系が知られている [1, 2]。また，アフィンワイル群対称性 $E_7^{(1)}$ 型 q-差分パンルヴェ方程式の多変数版も知られている [3]。しかし，これらの楕円差分ガルニエ系と，坂井の退化図式に現れる q-差分系の多変数版との関係は十分には明らかでない。

一方，坂井の分類において，離散パンルヴェ方程式はアフィンワイル群対称性の型により分類され，楕円差分を最上位とする退化図式をもつ。特に，楕円差分パンルヴェ方程式からアフィンワイル群対称性 $E_8^{(1)}$ 型 q-差分パンルヴェ方程式，さらに $E_7^{(1)}$ 型 q-差分パンルヴェ方程式への退化

$$e\text{-}E_8^{(1)} \longrightarrow q\text{-}E_8^{(1)} \longrightarrow q\text{-}E_7^{(1)}$$

はよく知られている。

本講演では，坂井の退化図式の多変数化という観点から，楕円差分ガルニエ系 [1] の退化を考察する。ここでいう多変数化とは，離散系においてシフト可能なパラメータが増えることに対応するものであり，微分方程式における Garnier 系と同様に，独立変数の増加と従属変数の増加が対応するものとみなす。従って，得られる系は従属変数 $2n$ 個をもつ高次元の離散ガルニエ型の系として記述される。

具体的には，楕円差分ガルニエ系を $M_n(P(e\text{-}E_8))$ と表し，その退化として得られる q-差分の系を $M_n(P(q\text{-}E_8))$ および $M_n(P(q\text{-}E_7))$ と書くことにより，

$$M_n(P(e\text{-}E_8)) \longrightarrow M_n(P(q\text{-}E_8)) \longrightarrow M_n(P(q\text{-}E_7)), \qquad \dim M_n = 2n,$$

という図式を考える。特に $n = 1$ の場合には，通常の 2 次元の離散パンルヴェ方程式

$$e\text{-}E_8^{(1)} \longrightarrow q\text{-}E_8^{(1)} \longrightarrow q\text{-}E_7^{(1)}$$

に対応する。

また，離散パンルヴェ方程式は双有理変換として記述される。本講演では，$e\text{-}E_8^{(1)}$ 型，$q\text{-}E_8^{(1)}$ 型，$q\text{-}E_7^{(1)}$ 型の多変数版として現れる離散ガルニエ型の系を，双有理変換として記述できるかを考察する。特に，楕円差分ガルニエ系から $p \to 0$ 極限によって得られる

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本研究は科研費（課題番号：JP23K03173）の助成を受けたものである。
2020 Mathematics Subject Classification: 33E17, 34M55, 39A13.
キーワード：elliptic Garnier system; discrete Painlevé equation; q-Garnier system; birational transformation; degeneration.

q-$E_8^{(1)}$ 型の多変数系，およびその退化として得られる q-$E_7^{(1)}$ 型の多変数系の間の関係を調べ，離散パンルヴェ方程式の退化図式と高次元の離散ガルニエ型の系との対応関係を整理する。

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Middle Laplace transform の q 類似と $A_4^{(1)}$ 型 q-Painlevé 方程式

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本講演の第一著者によって, Laplace 変換 $y(x) \mapsto \int y(t)e^{-xt}\,dt$ に付随する線形常微分方程式の変換として middle Laplace transform $\mathcal{ML}$ が導入された [1]. これは middle convolution [3] の Laplace 変換版で, 次の形の方程式全体の集合に閉じて作用する：

$$\frac{du}{dx} = \left(A_\infty + \sum_{i=1}^{m} \frac{A_i}{x-t_i}\right)u \qquad (A_\infty, A_i : n \times n \text{ 定数行列})$$

また Fuchs 型方程式に対する middle convolution mc_λ は $\mathcal{ML}$ を用いて表せる：

$$mc_\lambda = \mathcal{ML}^{-1} \circ add_{-\lambda} \circ \mathcal{ML}.$$

ここで $add_{-\lambda}$ は未知関数に対する gauge 変換 $u \mapsto x^{-\lambda}u$ に付随する方程式の変換である.

本稿の目的は, $\mathcal{ML}$ の q 差分類似を構成することである. 以降, $q \in \mathbb{C}$ を $0 < |q| < 1$ ととり固定する. q 差分商 $D_x : f(x) \mapsto \dfrac{f(x)-f(qx)}{x-qx}$ を用いた次の形の方程式を考える：

$$D_x u = \left(A_\infty + \sum_{i=1}^{m} \frac{A_i}{x-t_i}\right)u \qquad (A_\infty, A_i : n \times n \text{ 定数行列})$$

ただし, A_∞ は対角行列とする. この方程式に対して $U(x) := \left(\dfrac{u(x)}{x-t_1}, \dfrac{u(x)}{x-t_2}, \dots, \dfrac{u(x)}{x-t_m}\right)^{\mathrm{T}}$ とおけば q-Okubo 型と呼ばれる次の多項式係数の方程式が得られる：

$$(qx - S)D_x U = \left(A - I + A_\infty^{\oplus m}(x-S)\right)U$$

ここで I は単位行列, A, S, $A_\infty^{\oplus m}$ は A_i, t_i, A_∞ を並べてできる $nm \times nm$ 行列 (S は対角行列) である. 積分路 C をうまく取れば, $U(x)$ の Laplace 型 Jackson 積分変換

$$V(x) = \int_C U(s)(q(1-q)xs)_\infty\, d_q s \qquad \big((z)_\infty = (1-z)(1-qz)(1-q^2z)\cdots\big)$$

が満たす方程式を $D_x \mapsto x$, $x \mapsto -q^{-1}T_x^{-1}D_x = -q^{-1}D_x^-$ により求めることができる：

$$D_x^- V = q\left(-S + \sum_{j=1}^{l} \frac{B_j}{x-\tau_j}\right)V.$$

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2020 Mathematics Subject Classification: 39A13, 34M55
キーワード：q 差分方程式, q-Laplace 変換, q-middle convolution

ここで B_j は A_i から定まる行列, τ_j は A_∞ の相異なる固有値 $(1 \leq j \leq l)$ である．$\mathbb{C}^{mn}$ の部分空間 $\mathcal{K} = \mathrm{Ker} A_1 \oplus \mathrm{Ker} A_2 \oplus \cdots \oplus \mathrm{Ker} A_m$ は S と各 B_j の作用で不変である．商空間 $\mathbb{C}^{mn}/\mathcal{K}$ への S, B_j の作用を $\bar{S}$, $\bar{B}_j$ としたとき, q 差分方程式の変換

$$D_x u = \left(A_\infty + \sum_{i=1}^{m} \frac{A_i}{x - t_i} \right) u \quad \mapsto \quad D_x^- v = q \left(-\bar{S} + \sum_{j=1}^{l} \frac{\bar{B}_j}{x - \tau_j} \right) v$$

を q-middle Laplace transform と呼び, $\mathcal{ML}_q$ と書く．$\mathcal{ML}_q$ については,

- 逆変換 $\mathcal{ML}_q^-$ の構成
- 既約性の保存
- q-middle convolution [2, 5] の $\mathcal{ML}_q$ を用いた分解

などがわかっている．

$\mathcal{ML}_q$ を $A_4^{(1)}$ 型 q-Painlevé 方程式の Lax 方程式 [4] へ適用してみよう．$A_4^{(1)}$ 型 q-Painlevé 方程式は次の 2 階の線形 q 差分方程式の適当な変形で得られる：

$$u(qx) = A(x)u(x),\ A(x) = A_0 + A_1 x + A_2 x^2,$$
$$A_0 \sim \begin{pmatrix} \theta_1 t & 0 \\ 0 & \theta_2 t \end{pmatrix},\ A_2 = \begin{pmatrix} \kappa_1 & 0 \\ 0 & 0 \end{pmatrix},\ \det A(x) = \kappa_1 \kappa_2 (x - a_1 t)(x - a_2 t)(x - a_3).$$

$\tilde{u}(x) = x^\rho (x/a_1)_\infty u(x)$ $(q^\rho = \theta_1^{-1})$ とし, $\tilde{u}$ の満たす方程式に $\mathcal{ML}_q$ を施して整理すれば, 次の方程式を得る：

$$v(qx) = \bar{A}(x)v(x),\ A(x) = \bar{A}_0 + \bar{A}_1 x + \bar{A}_2 x^2,$$
$$\bar{A}_0 \sim \begin{pmatrix} a_2 a_3 \kappa_1 t & 0 \\ 0 & \theta_2 t \end{pmatrix},\ \bar{A}_2 = \begin{pmatrix} \theta_1/(a_2 a_3) & 0 \\ 0 & 0 \end{pmatrix},\ \det \bar{A}(x) = \kappa_1 \kappa_2 \left(x + \frac{\theta_2}{\kappa_2} t \right) (x - a_2 t)(x - a_3).$$

つまり, $\mathcal{ML}_q$ により Lax 方程式に対する次の変換を与えたことになる：

$$a_1 \mapsto -\frac{\theta_2}{\kappa_2},\ \kappa_1 \mapsto \frac{\theta_1}{a_2 a_3},\ \kappa_2 \mapsto -\frac{\theta_2}{a_1},\ \theta_1 \mapsto a_2 a_3 \kappa_1.$$

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Three-term relations and some specializations of the μ function

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概　要

(一般化) μ 函数 [ST] の Al-Salam-Ismail 多項式 [AI] を用いた三項関係式の明示公式を導出する. その応用として特殊化の明示公式を与える.

q を $|q|<1$ なる複素数とし, q階乗積とテータ函数を

$$(x)_\infty = (x;q)_\infty := \prod_{j=0}^{\infty}(1-xq^j), \quad (x)_n = (x;q)_n := \frac{(x;q)_\infty}{(q^n x;q)_\infty},$$
$$\theta(x) = \theta(x;q) := (x;q)_\infty (q/x;q)_\infty,$$

で定める.

定義 1. x, y, a を $xa^{-1}, y \in \mathbb{C} \setminus q^{\mathbb{Z}}$ となる複素数とするとき, (一般化) μ 函数を以下のように定める [ST] :

$$\mu(x,y;a) := -iq^{-\frac{1}{8}}\left(\frac{x}{y}\right)^{\frac{\alpha}{2}} \frac{(x)_\infty}{(q,x/a)_\infty \theta(y)} \sum_{n\in\mathbb{Z}} (-1)^n q^{\binom{n}{2}} \frac{(x/a)_n}{(x)_n} y^n.$$

更に対称化された μ 函数を

$$\mathrm{M}(x,y;a) := iq^{\frac{1}{8}}\left(\frac{y}{x}\right)^{\frac{a}{2}} \frac{\theta(x/a)}{\theta(x)}\mu(x,y;a) = \frac{(qa/x)_\infty}{(q,q/x)_\infty \theta(y)} \sum_{n\in\mathbb{Z}} (-1)^n q^{\binom{n}{2}} \frac{(x/a)_n}{(x)_n} y^n$$

とおく.

命題 2. (1) **差分方程式**: $T_{q,x}$ を x 変数の q シフト作用素とすると,

$$\left[T_{q,x}^2 - \left(1-q\frac{x}{y}\right)aT_{q,x} - aq\frac{x}{y}\right]\mathrm{M}(x,y;a) = 0,$$
$$\left[T_{q,x}^{-1} - 1\right]\mathrm{M}(x,y;a) = (a-1)\frac{q}{x}M(x,y;qa).$$

(2) **対称性**:

$$\mathrm{M}(x,y;a) = \mathrm{M}(y,x;a),$$
$$\mathrm{M}(q^{\pm 1}x, q^{\pm 1}y;a) = \mathrm{M}(x,y;a)a^{\pm 1}.$$

(3) **特殊化**:

$$\mathrm{M}(x,-x;a) = \frac{(aq;q^2)_\infty}{(q;q^2)_\infty}\frac{\theta(x^2/a;q^2)}{\theta(x^2;q^2)},$$

$$\mathrm{M}(x,q^{\frac{1}{2}}x;a) = \frac{(q^{\frac{1}{2}};q)_\infty}{2\theta(q^{\frac{1}{2}}x)\theta(x)}\{(a^{\frac{1}{2}};q^{\frac{1}{2}})_\infty \theta(x/a^{\frac{1}{2}})\theta(q^{\frac{1}{2}}x/a^{\frac{1}{2}}) + (-a^{\frac{1}{2}};q^{\frac{1}{2}})_\infty \theta(-x/a^{\frac{1}{2}})\theta(-q^{\frac{1}{2}}x/a^{\frac{1}{2}})\}.$$

本研究は科研費 (JSPS KAKENHI Grant Number 26K06837) の助成を受けたものである.
キーワード：q-hypergeometric series, μ function, Al-Salam-Ismail polynomials, difference equations
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数列 $\{S_n(\alpha,\beta,\gamma;q)\}_n$ と $\{T_{n-1}(\alpha,\beta,\gamma;q)\}_n$ を漸化式

$$V_{n+1}(\alpha,\beta,\gamma;q)=\gamma(1+\alpha q^n)V_n(\alpha,\beta,\gamma;q)-\beta q^{n-1}V_{n-1}(\alpha,\beta,\gamma;q)$$

を満たし, それぞれ初期値が

$$\begin{aligned} S_0(\alpha,\beta,\gamma;q)&=0, \quad S_1(\alpha,\beta,\gamma;q)=1,\\ T_{-1}(\alpha,\beta,\gamma;q)&=1, \quad T_0(\alpha,\beta,\gamma;q)=0 \end{aligned}$$

によって定めるとする.

命題 3. n を非負整数とするとき,

$$S_n(\alpha,\beta,\gamma;q)=U_{n-1}(q\alpha,q\beta,\gamma;q), \quad T_{n-1}(\alpha,\beta,\gamma;q)=-\beta U_{n-2}(q^2\alpha,q^2\beta,\gamma;q).$$

ただし, $U_n(\alpha,\beta,\gamma;q)$ は Al-Salam-Ismail 多項式 [AI]:

$$\begin{aligned} U_n(\alpha,\beta,\gamma;q)&:=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-\alpha)_{n-k}(q)_{n-k}(-\beta)^k}{(-\alpha)_k(q)_k(q)_{n-2k}}q^{k(k-1)}\gamma^{n-2k},\\ U_{-1}(\alpha,\beta,\gamma;q)&:=0, \quad U_{-2}(\alpha,\beta,\gamma;q):=-\frac{q}{\beta}. \end{aligned}$$

定理 4 (三項間関係式)**.** 任意の非負整数 n について,

$$\begin{aligned} \mathrm{M}(q^{n-1}x,y;a)&=\mathrm{M}(y,q^{n-1}x;a)=\mathrm{M}(x,q^{1-n}y;a)a^{n-1}=\mathrm{M}(q^{1-n}y,x;a)a^{n-1}\\ &=t_{n-1}(x,y;a)\mathrm{M}(q^{-1}x,y;a)+s_n(x,y;a)\mathrm{M}(x,y;a)\\ &=\{t_{n-1}(x,y;a)+s_n(x,y;a)\}\mathrm{M}(x,y;a)\\ &\quad+(a-1)\frac{q}{x}t_{n-1}(x,y;a)\mathrm{M}(x,y;qa). \end{aligned}$$

ただし,

$$s_n(x,y;a):=S_n\left(-\frac{x}{y},-\frac{aqx}{y},a;q\right), \quad t_{n-1}(x,y;a):=T_{n-1}\left(-\frac{x}{y},-\frac{aqx}{y},a;q\right).$$

系 5.

$$\begin{aligned} \mathrm{M}(q^{n-1}x,-x;a)&=\mathrm{M}(-x,q^{n-1}x;a)=\mathrm{M}(x,-q^{1-n}x;a)a^{n-1}=\mathrm{M}(-q^{1-n}x,x;a)a^{n-1}\\ &=\{T_{n-1}(1,aq,a;q)+S_n(1,aq,a;q)\}\frac{(aq;q^2)_\infty}{(q;q^2)_\infty}\frac{\theta(x^2/a;q^2)}{\theta(x^2;q^2)}\\ &\quad+(a-1)\frac{q}{x}T_{n-1}(1,aq,a;q)\frac{(aq^2;q^2)_\infty}{(q;q^2)_\infty}\frac{\theta(x^2/aq;q^2)}{\theta(x^2;q^2)}. \end{aligned}$$

$\mathrm{M}(q^{n-1}x,q^{\frac{1}{2}}x;a)$ についても同様の公式が導出できる.

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Algebras and Bispectrality

a spiral from exactly solvable models to special functions

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Abstract

The exact solvability of many models is governed by symmetries, which are frequently not manifest. These symmetries are encoded in algebraic structures whose representations yield the special functions in terms of which the solutions are expressed. This talk approaches this circle of ideas through the notion of *bispectrality*: a family of functions satisfying an eigenvalue equation in each of two dual variables. I shall outline the algebraic theory of bispectrality, describe the associated algebras—Racah, Askey–Wilson, Bannai–Ito, together with their higher-rank and rational extensions—explain how their representations produce the classical and q-orthogonal polynomials as overlaps of eigenbases, and discuss applications ranging from association schemes and superintegrable models to the entanglement of free fermions. The lecture closes at the current frontier, where orthogonal polynomials give way to rational functions.

1 Bispectrality

Certain families of special functions share a common structural feature: their member functions $\psi_n(x)$, depending on the two (possibly discrete) variables n and x, satisfy an eigenvalue equation in each of them. Namely, one has

$$A\,\psi_n(x) = \Lambda(n)\,\psi_n(x),$$

and,

$$A^*\,\psi_n(x) = \theta(x)\,\psi_n(x).$$

This property is called *bispectrality.* Its significance depends on the class of operators considered: requiring A and A^* to be of bounded order—for instance, A and A^* second-order differential or difference operators—makes bispectrality a strong constraint, whose solutions can be classified (Duistermaat–Grünbaum). For such pairs the two operators generally do not commute, yet they close into a finite quadratic algebra whose representation theory largely characterizes $\psi_n(x)$. The aim of this lecture is to describe the algebraic underpinning of bispectrality. A convenient finite-dimensional formulation is due to Terwilliger. An ordered pair (A, A^*) of operators on a finite-dimensional space V is a *Leonard pair* if there is a basis in which A is diagonal and A^* is irreducible tridiagonal, *and* a basis in which A^* is diagonal and A is irreducible tridiagonal (all sub- and super-diagonal entries nonzero). Each operator is thus tridiagonal in an eigenbasis of the other, and the overlaps between the two bases are bispectral functions. Two operators that are simultaneously diagonal in a single basis merely

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2020 Mathematics Subject Classification: 33D45, 33C45, 16S99, 81R12, 05E30.
Keywords: bispectrality, Leonard pairs, Askey–Wilson algebra, Racah algebra, Bannai–Ito algebra, orthogonal polynomials, association schemes, algebraic Heun operator, rational functions.

commute, $[A, A^*] = 0$. It is striking that passing to the next order of complexity—the one stated in the Leonard-pair definition—forces A and A^* to generate a quadratic algebra. Writing the q-commutator,

$$[X, Y]_q := qXY - q^{-1}YX \qquad \big([X, Y]_q \xrightarrow{q\to 1} [X, Y]\big),$$

the pair is then found to obey the defining relations of the Askey–Wilson algebra that involve this q-commutator and that are given below. The lecture is arranged as a spiral: a single object—the quadratic algebra of a bispectral pair—returns at successively wider levels. It first yields the classical orthogonal polynomials; it then reappears as a symmetry algebra in combinatorics, in the representation theory of Lie (super)algebras, and in exactly solvable models of physics; and it is further carried to $q = -1$, to higher rank, and beyond polynomials to rational functions. These realizations are largely independent, so each can be followed on its own.

2 Overlaps and orthogonal polynomials

The functions attached to a bispectral pair arise as overlaps of its two eigenbases. Take a self-adjoint pair with orthonormal eigenbases $|\theta_k\rangle$ and $|\theta^*_m\rangle$, and set $\phi(k, m) = \langle\theta_k\,|\theta^*_m\rangle$. In the matrix element $\langle\theta_k|\, A^*\,|\theta^*_m\rangle$, letting A^* act to the right returns $\theta^*_m\,\phi(k, m)$, while letting it act to the left—where it is tridiagonal in the eigenbasis of A—yields a three-term recurrence in k,

$$\theta^*_m\,\phi(k, m) = \beta_k\,\phi(k{+}1, m) + \alpha_k\,\phi(k, m) + \gamma_k\,\phi(k{-}1, m),$$

and the element $\langle\theta_k|\, A\,|\theta^*_m\rangle$ gives, in the same way, a recurrence in m. The overlap is therefore bispectral in the two discrete variables—exactly the structure of a classical orthogonal polynomial, here in a discrete variable. Indeed, let $\{P_n(x)\}$ be monic orthogonal polynomials, $\deg P_n = n$. Favard's theorem provides the first eigenvalue problem, the three-term recurrence

$$x\,P_n(x) = P_{n+1}(x) + b_nP_n(x) + u_nP_{n-1}(x),$$

and "classical" means precisely that a second one holds, a difference or differential equation $D\,P_n(x) = \lambda_nP_n(x)$ acting on x. The classical polynomials are bispectral, and they sit at the confluence of the two dual recurrences above. At the top of the discrete ($q = 1$) Askey scheme stand the *Racah polynomials*, of degree n in the quadratic grid $\lambda(x) = x(x{+}\gamma{+}\delta{+}1)$,

$$R_n\big(\lambda(x)\big) = {}_4F_3\left(\begin{matrix}-n,\ n{+}\alpha{+}\beta{+}1,\ -x,\ x{+}\gamma{+}\delta{+}1\\ \alpha{+}1,\ \beta{+}\delta{+}1,\ \gamma{+}1\end{matrix}\,;\,1\right).$$

They satisfy a recurrence in n (eigenvalue $\lambda(x)$) and a second-order difference equation in x (eigenvalue $n(n{+}\alpha{+}\beta{+}1)$). The Racah duality $x \leftrightarrow n$, $\alpha \leftrightarrow \gamma$, $\beta \leftrightarrow \delta$ interchanges them. Denoting the two bispectral operators K_1, K_2 and setting $K_3 = [K_1, K_2]$, they close into the *Racah algebra*

$$\begin{aligned}
[K_1, K_2] &= K_3,\\
[K_2, K_3] &= K_2^2 + \{K_1, K_2\} + d\,K_2 + e_1,\\
[K_3, K_1] &= K_1^2 + \{K_1, K_2\} + d\,K_1 + e_2,
\end{aligned}$$

whose structure constants d, e_1, e_2 are explicit polynomials in $\alpha, \beta, \gamma, \delta$. On the linear grid one contracts to the Hahn polynomials and an analogous Hahn algebra.

3 Leonard's theorem

These examples are not isolated: a classification theorem shows them to be essentially exhaustive. Leonard's theorem (1982; the association-scheme account is Bannai–Ito 1984, the linear-algebraic form Terwilliger 2001) states that a finite family of orthogonal polynomials whose dual is *again* orthogonal must be a special or limiting case of the (q-)Racah/Askey–Wilson polynomials. In Leonard-pair language the two spectra are tightly constrained: there is a single scalar $\beta = q + q^{-1}$ with

$$\theta_{i-1} - \beta\,\theta_i + \theta_{i+1} = \gamma, \qquad \theta^*_{i-1} - \beta\,\theta^*_i + \theta^*_{i+1} = \gamma^*,$$

so each spectrum lies on a q-quadratic grid $\theta_i = a + b\,q^i + c\,q^{-i}$, and the overlaps $\langle \theta_i \,|\theta^*_j\rangle$ are q-Racah polynomials. This gives a triple correspondence central to the theory,

$$\text{Leonard pair } (A, A^*) \iff \text{Askey–Wilson algebra} \iff (q\text{-})\text{Racah polynomials}.$$

4 The Askey–Wilson algebra

Iterating the q-brackets of (A, A^*) with $K_1 = A$, $K_2 = A^*$, $K_3 = [K_1, K_2]_q$ closes the *Askey–Wilson algebra*

$$\begin{aligned}
[K_1, K_2]_q &= K_3,\\
[K_2, K_3]_q &= K_2^2 + \{K_1, K_2\} + d\,K_2 + e_1,\\
[K_3, K_1]_q &= K_1^2 + \{K_1, K_2\} + d\,K_1 + e_2,
\end{aligned}$$

the quadratic algebra of the Leonard pair and the q-deformation of the Racah algebra of Section 2. Its finite-dimensional representations reproduce, as overlaps, the full q-Racah tableau.

5 Association schemes

This section is an entry point for the combinatorialist. Recall only that a bispectral pair is a couple of operators, each tridiagonal in an eigenbasis of the other; such pairs are already present inside finite combinatorial objects. A symmetric *association scheme* on a finite set X is a family of $\{0,1\}$ adjacency matrices $A_0, \dots, A_D$ with

$$A_0 = I, \qquad \sum_{i=0}^{D} A_i = J, \qquad A_iA_j = \sum_{k=0}^{D} p^k_{ij} A_k = A_jA_i,$$

spanning the commutative *Bose–Mesner algebra*—the matrices the scheme diagonalizes simultaneously—with primitive idempotents $E_0, \dots, E_D$ projecting onto the common eigenspaces. When both families are polynomial in a single generator (the scheme is P- and Q-polynomial), one has an adjacency matrix A and a dual adjacency matrix A^*; adjoining them gives the noncommutative *Terwilliger algebra* $T = \langle A, A^*\rangle$, on each irreducible module of which (A, A^*) is a Leonard pair. The eigenvalue data of the scheme are then (q-)Racah-type polynomials, so the combinatorics is governed by the algebra of Section 4. The Johnson scheme $J(n, k)$—k-subsets of $\{1, \dots, n\}$, adjacent when they share $k-1$ elements—is the canonical illustration. Its Terwilliger algebra is a central extension of the Hahn algebra [9]: writing a k-subset as a spin configuration, the dual adjacency $A^*_J = \sum_i \sigma^{(i)}_z = 2J^z$ (with $\sigma_z = \mathrm{diag}(1, -1)$ the Pauli matrix) and a Johnson edge is a weight-preserving double spin flip, so A_J is essentially the total $\mathfrak{su}(2)$ Casimir. The change between the coupled and uncoupled bases is the $\mathfrak{su}(2)$ Clebsch–Gordan transform, with dual Hahn coefficients, so that the combinatorial and representation-theoretic descriptions coincide.

6 Recoupling and superintegrability

The Racah algebra—the $q \to 1$ quadratic algebra of Sections 2 and 4—also governs the recoupling of three copies of $\mathfrak{su}(1,1)$ in the quantum theory of angular momentum. With Casimirs $C^{(i)} = \lambda_i$ and intermediate Casimirs $C^{(ij)}$, the identity

$$C^{(123)} = C^{(12)} + C^{(23)} + C^{(31)} - C^{(1)} - C^{(2)} - C^{(3)}$$

leaves two independent intermediate Casimirs, and $K_1 = -\frac{1}{2}C^{(12)}$, $K_2 = -\frac{1}{2}C^{(23)}$ generate the Racah algebra. The recoupling ($6j$) coefficients are the Racah polynomials [1]. The very same algebra is the symmetry algebra of the generic three-parameter superintegrable model on the two-sphere,

$$H = J_1^2 + J_2^2 + J_3^2 + \frac{k_1^2 - \frac{1}{4}}{x_1^2} + \frac{k_2^2 - \frac{1}{4}}{x_2^2} + \frac{k_3^2 - \frac{1}{4}}{x_3^2}, \qquad x_1^2 + x_2^2 + x_3^2 = 1.$$

Here $J_i = -\mathrm{i}\,\epsilon_{ijk} x_j \partial_k$ (with summation over j, k). Realizing the three $\mathfrak{su}(1,1)$ by singular oscillators (radial motion in an inverse-square potential) gives $H = 4\,C^{(123)} + \frac{3}{4}$, and each constant of motion is an intermediate Casimir, $L_k = 4\,C^{(ij)} - a_i - a_j + 1$. Recoupling and superintegrability are thus governed by the same algebra.

7 The Bannai–Ito algebra and higher rank

Leonard's classification has a second edge. Alongside the limit $q \to 1$ (Racah) sits $q \to -1$, where the polynomials become the *Bannai–Ito* family [2]. These are bispectral through a first-order Dunkl shift operator—a difference operator built from the reflection $Rf(x) = f(-x)$, in the manner of the Dunkl calculus of reflection groups—

$$L = F(x)(1 - R) + G(x)(T^+R - 1), \qquad L\,P_n = \lambda_n P_n,$$

and L together with multiplication by x generates, through anticommutators, the *Bannai–Ito algebra*

$$\{K_1, K_2\} = K_3 + \omega_3, \quad \{K_2, K_3\} = K_1 + \omega_1, \quad \{K_3, K_1\} = K_2 + \omega_2,$$

whose natural home is the superalgebra $\mathfrak{osp}(1|2)$. It is the exact $q \to -1$ mirror of the Racah construction, with three $\mathfrak{osp}(1|2)$ replacing three $\mathfrak{su}(1,1)$. A more recent and active direction couples n copies rather than three, producing the *higher-rank* Racah algebra $\mathrm{Rac}(n)$ [3], generated by the intermediate Casimirs C_A over subsets $A \subseteq [n]$. A central quotient of $\mathrm{Rac}(n)$ is precisely the centralizer of the diagonal $\mathfrak{su}(2)$ in $U(\mathfrak{su}(2))^{\otimes n}$ [4]. Its overlaps are the multivariate (Tratnik) Racah polynomials, and its combinatorial homes are the multivariate P-polynomial association schemes and the associated m-distance-regular graphs that we have recently introduced [5].

8 The algebraic Heun operator

Until now the two eigenbases of a bispectral pair have been related only through their overlaps; there is also a particular operator that couples them, and it turns out to arise in a range of concrete problems. Given any bispectral pair (X, Y), the bilinear

$$W = \tau_1 XY + \tau_2 YX + \tau_3 X + \tau_4 Y + \tau_0 I$$

is, for a Leonard pair, the unique operator that is tridiagonal in *both* eigenbases (Nomura–Terwilliger). For the Jacobi pair, with $X = x$ and Y the hypergeometric operator, W is the classical Heun operator—hence the name *algebraic Heun operator* [6]—and it is also characterized as the most general operator that raises every polynomial degree by one. Such an operator is associated to every bispectral algebra: Racah, Askey–Wilson, Bannai–Ito. This operator has concrete uses in areas far from special-function theory. It underlies the *time–band limiting* problem of Slepian, Landau and Pollak, a cornerstone of signal analysis: a local second-order operator commuting with the pair of limiting projections—which are difficult to diagonalize numerically—is obtained from the algebraic Heun operator of the underlying bispectral pair. The same mechanism enters into the determination of the entanglement entropy of free fermions [7], a question of great significance in quantum information. The ground-state correlation matrix

$$C_{mn} = \langle\Psi_0|\, c_m^\dagger c_n \,|\Psi_0\rangle\,,$$

restricted to a subsystem, is a full matrix with nearly degenerate eigenvalues; but when the single-particle wavefunctions obey a recurrence with Askey-scheme coefficients, the chopped C commutes with a tridiagonal algebraic Heun operator sharing its eigenbasis. On distance-regular graphs (e.g. the Hamming [8] and Johnson [9] schemes) the chopped C lives in the Terwilliger algebra $\mathcal{T}$, and the same tool, together with the representation theory of $\mathcal{T}$, allows the diagonalization of the entanglement Hamiltonian block by block.

9 Rational functions

The last part of the lecture turns to the current frontier of the subject. Orthogonal polynomials solve an ordinary eigenvalue problem (EVP); *biorthogonal rational functions* solve a generalized one (a generalized eigenvalue problem, GEVP), $L\,\varphi_n = \lambda_n M\,\varphi_n$ for two tridiagonal operators L, M (Zhedanov). To treat both problems in a unified way, *meta algebras* have been introduced [10]. EVPs and GEVPs are posited in modules of these algebras and overlaps between EVP and GEVP bases are found to be bispectral rational functions. For the Hahn case the *meta-Hahn algebra*

$$[Z, X] = Z^2 + Z, \quad [V, Z] = 2X + \eta_1 Z + \eta_3 I, \quad [X, V] = \{V, Z\} + \eta_1 X + V - \eta_1 Z + \eta_0 I$$

contains the Hahn algebra as a Leonard pair and returns both the Hahn polynomials and the Hahn rational functions (both of ${}_3F_2$ type). The most recent development recasts this in linear-algebraic terms. A *Leonard trio* $(V, \widetilde{V}, Z)$ [11] involves the existence of eigenbases for V and $\widetilde{V}$ and the property that Z and products of Z with V or $\widetilde{V}$ be tridiagonal in these bases. The overlaps of these bases satisfy two three-term generalized eigenvalue problems and are therefore bispectral rational functions. Where a Leonard *pair* produces polynomials, a Leonard *trio* produces rational functions; setting $Z = I$ collapses the trio to a pair. In the Hahn case the trio algebra is isomorphic to the meta-Hahn algebra, and in the Wilson case the overlaps are the Wilson rational functions. Classifying irreducible Leonard trios is the rational analogue of classifying Leonard pairs. This task would eventually provide a *rational Askey scheme.*

10 Conclusion

A central theme runs through the material presented. Bispectrality—the property of a family of functions satisfying both a recurrence and a difference or differential equation—is encoded in a quadratic algebra whose representations return those functions as overlaps of eigenbases. On the polynomial side one has Leonard pairs and the Racah/Askey–Wilson algebra, realized in association schemes, in $\mathfrak{su}(1,1)$ recoupling and in superintegrable models. For $q \to -1$ one finds the Bannai–Ito algebra via $\mathfrak{osp}(1|2)$, and higher rank considerations connect to multivariate polynomials. The algebraic Heun operator is tridiagonal in both eigenbases and provides a means of characterizing free-fermion entanglement. On the rational side, generalized eigenvalue problems, meta algebras and Leonard trios provide the algebraic framework, with the Wilson rational functions at the top of the scheme. Several questions remain open—the full classification of Leonard trios, the rational theory at $q \to -1$, multivariate rational functions, and higher-rank meta algebras. In each case considered, the solvable structure is governed by a quadratic algebra whose eigenbasis overlaps are the associated special functions.

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