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Grothendieck 多項式の特殊値と超幾何函数との関係

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Grothendieck 多項式は，Schur 多項式の K 理論版として Lascoux–Schützenberger [4] により定義された．Schur 多項式に対する Weyl の恒等式の一般化が Ikeda–Naruse [3] により示されており，本講演ではこれを Grothendieck 多項式の定義とする．論文 [1] に基づき，その特殊値と Gauss および Holman の超幾何函数との関係を紹介する．

非負整数列 $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ を非負整数の分割とする．論文 [3] により，Grothendieck 多項式 $G_\lambda(x \mid \beta)$ は次の表示をもつ:

$$G_\lambda(x \mid \beta) = \frac{\left| x_i^{\lambda_j+n-j}(1+\beta x_i)^{j-1} \right|_{n\times n}}{\prod_{1\le i<j\le n}(x_i - x_j)}. \tag{1}$$

ここで，変数 $x \in \mathbb{C}^n$ およびパラメータ $\beta \in \mathbb{C}$ とする．上式において $\beta = 0$ とすると，Grothendieck 多項式は Schur 多項式に一致する．即ち，$G_\lambda(x \mid 0) = s_\lambda(x)$ である．

式 (1) において，Schur 多項式のフック長公式および Lenart [5] による Grothendieck 多項式の Schur 多項式への展開を用いると，次が得られる．

Proposition 1 ([1])**.** 分割 $\lambda = (k, 0, \ldots, 0)$ および $\lambda = (1^k, 0, \ldots, 0)$ $(k \in \mathbb{Z}_{>0})$ に対して，それぞれ次が成り立つ．

$$(1)\ G_{(k)}(1,1,\ldots,1 \mid \beta) = \binom{n+k-1}{k} {}_2F_1\left(\begin{matrix} k,\ 1-n \\ k+1 \end{matrix}; -\beta\right).$$

$$(2)\ G_{(1^k)}(1,1,\ldots,1 \mid \beta) = \binom{n}{k} {}_2F_1\left(\begin{matrix} k,\ k-n \\ k+1 \end{matrix}; -\beta\right).$$

式 (1) に対し，変数の主特殊化 $(x_1, \ldots, x_n) = (1, q, q^2, \ldots, q^{n-1})$ $(q \in \mathbb{C}, |q| < 1)$ を行い，行列式の列線型性から次が得られる．

Theorem 2 ([1])**.** 任意の分割 λ に対して，次が成り立つ．

$$\begin{aligned} G_\lambda(1, q, q^2, \ldots, q^{n-1} \mid \beta) &= \sum_{k_1=0}^{0}\sum_{k_2=0}^{1}\cdots\sum_{k_n=0}^{n-1}\binom{0}{k_1}\binom{1}{k_2}\cdots\binom{n-1}{k_n}\beta^{k_1+\cdots+k_n} \\ &\quad\times \prod_{1\le i<j\le n}\frac{q^{\lambda_j+n-j+k_j} - q^{\lambda_i+n-i+k_i}}{q^{n-j} - q^{n-i}}. \end{aligned} \tag{2}$$

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式 (2) において $q \to 1$ の極限と $\beta = 1$ の特殊化を考えると，次の集合値半標準盤の個数 $|\mathrm{SVT}(\lambda, n)|$ の明示公式が得られる．

Corollary 3 ([1])**.** 任意の分割 λ に対して，次が成り立つ．

$$|\mathrm{SVT}(\lambda, n)| = \sum_{k_1=0}^{0} \sum_{k_2=0}^{1} \cdots \sum_{k_n=0}^{n-1} \binom{0}{k_1}\binom{1}{k_2}\cdots\binom{n-1}{k_n} \prod_{1 \le i < j \le n} \frac{\lambda_i - \lambda_j + k_i - k_j + j - i}{j - i}.$$

他方，式 (2) は以下の Holman の超幾何函数 $F^{(n)}$ とよばれる多重超幾何函数 [2] を用いて表すことができる:

$$\begin{aligned} &F^{(n)}((A_{ij})_{(n-1)\times(n-1)} | (a_{ij})_{n \times u} | (b_{ij})_{n \times v} | (z_{i1})_{n \times 1}) \\ &= \sum_{k_1, \ldots, k_n = 0}^{\infty} \left(\prod_{1 \le i < j \le n} \frac{A_{ij} + k_i - k_j}{A_{ij}} \right) \left(\prod_{j=1}^{u} \prod_{i=1}^{n} (a_{ij})_{k_i} \right) \left(\prod_{j=1}^{v} \prod_{i=1}^{n} \frac{1}{(b_{ij})_{k_i}} \right) \left(\prod_{i=1}^{n} z_{i1}^{k_i} \right), \end{aligned}$$

ここで，行列 $(A_{ij})_{(n-1)\times(n-1)}$ は次の通りである．

$$(A_{ij})_{(n-1)\times(n-1)} = \begin{pmatrix} A_{12} & & & \\ A_{13} & A_{23} & & \huge 0 \\ \vdots & \vdots & \ddots & \\ A_{1n} & A_{2n} & \cdots & A_{n-1,n} \end{pmatrix}.$$

Holman の超幾何函数は，ユニタリ群 $U(n+1)$ の表現論に現れることで知られている．

Theorem 4 ([1])**.** 任意の分割 λ に対して，半標準盤の個数 $|\mathrm{SST}(\lambda, n)|$ を用いると，次が成り立つ．

$$\begin{aligned} &G_\lambda(1, 1, \ldots, 1 \mid \beta) \\ &= |\mathrm{SST}(\lambda, n)| \times F^{(n)} \left(\begin{pmatrix} A_{12} & & & \\ A_{13} & A_{23} & & \huge 0 \\ \vdots & \vdots & \ddots & \\ A_{1n} & A_{2n} & \cdots & A_{n-1,n} \end{pmatrix} \middle| \begin{pmatrix} 0 \\ -1 \\ \vdots \\ -n+1 \end{pmatrix} \middle| \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \middle| \begin{pmatrix} -\beta \\ -\beta \\ \vdots \\ -\beta \end{pmatrix} \right). \end{aligned}$$

ここで，$A_{ij} = \lambda_i - \lambda_j + j - i$ とする．

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Virasoro Action on Schur Q-Functions and Pfaffian Identities

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1 Schur Q-Functions and Their Definitions

Schur Q-functions were originally introduced by Schur in the context of projective representations of symmetric groups. We begin by reviewing their definitions and fundamental properties.

Let a partition be a finite sequence of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$. The size of λ is $|\lambda| = \sum_{i=1}^{\ell} \lambda_i$, and its length is denoted by $\ell(\lambda)$. A partition with distinct parts is called a *strict partition*, and we write $\mathcal{SP}(n)$ for the set of strict partitions of n. Let $V = \mathbb{C}[t_j : j \geq 1, \text{ odd}]$ be the polynomial ring graded by $\deg t_j = j$, and write $V = \bigoplus_{n=0}^{\infty} V(n)$, where $V(n)$ denotes the space of homogeneous polynomials of degree n. Define an inner product on V by $\langle F, G\rangle = F(2\widetilde{\partial})\,\overline{G}(t)\big|_{t=0}$, where $2\widetilde{\partial} = \left(2\partial_1, \frac{2}{3}\partial_3, \frac{2}{5}\partial_5, \ldots\right)$ with $\partial_j = \frac{\partial}{\partial t_j}$.

Put $\xi(t,u) = \sum_{j\geq 1,\text{ odd}} t_j u^j$ and define $q_n(t) \in V(n)$ by

$$e^{\xi(t,u)} = \sum_{n=0}^{\infty} q_n(t) u^n.$$

For integers a, b with $a > b > 0$, define

$$Q_{ab}(t) := q_a(t)q_b(t) + 2\sum_{i=1}^{b}(-1)^i q_{a+i}(t)q_{b-i}(t), \quad Q_{ba}(t) := -Q_{ab}(t).$$

Given a strict partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{2m})$, the associated Schur Q-function is defined by the Pfaffian

$$Q_\lambda(t) := \mathrm{Pf}\left(Q_{\lambda_i\lambda_j}(t)\right)_{1\leq i,j\leq 2m}.$$

The Schur Q-function $Q_\lambda(t)$ is homogeneous of degree $|\lambda|$, and the collection $\{Q_\lambda(t) \mid |\lambda| = n\}$, indexed by strict partitions, forms an orthogonal basis for $V(n)$ with respect to the above inner product.

2 Quadratic Relations for Schur Q-Functions

For $\lambda = (\lambda_1, \ldots, \lambda_{2m}) \in \mathcal{SP}$, Schur Q-functions satisfy the following quadratic relations:

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Theorem 1. ([1])
For $\lambda = (\lambda_1, \dots, \lambda_{2m})$,

$$\sum_{i=2}^{2m}(-1)^i \partial_1 Q_{\lambda_1,\lambda_i}\partial_1 Q_{\lambda_2,\dots,\widehat{\lambda}_i,\dots,\lambda_{2m}} = 0,$$
$$\sum_{i=2}^{2m}(-1)^i \left(\partial_1 Q_{\lambda_1,\lambda_i}\partial_3 Q_{\lambda_2,\dots,\widehat{\lambda}_i,\dots,\lambda_{2m}} + \partial_3 Q_{\lambda_1,\lambda_i}\partial_1 Q_{\lambda_2,\dots,\widehat{\lambda}_i,\dots,\lambda_{2m}}\right) = 0.$$

When $m = 2$, the first identity in Theorem 1 coincides with the classical Plücker relation that characterizes the Grassmannian as a projective variety. This observation suggests the possibility of a geometric interpretation underlying the formulas in Theorem 1.

3 Virasoro Operators and Main Results

For a positive odd integer j, put $a_j = \sqrt{2}\partial_j$ and $a_{-j} = \frac{j}{\sqrt{2}} t_j$ so that they satisfy the Heisenberg relation as operators on V :

$$[a_j, a_i] = j\delta_{j+i,0}.$$

For an integer k, put

$$L_k = \frac{1}{2}\sum_{j\in\mathbb{Z}_{\text{odd}}} : a_{-j}a_{j+2k} : + \frac{1}{8}\delta_{k,0},$$

where

$$: a_j a_i := \begin{cases} a_j a_i & \text{if } j \le i, \\ a_i a_j & \text{if } j > i \end{cases}$$

is the normal ordering.

Theorem 2. ([1])
Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{2m})$ be a strict partition. Then for any $k \ge 1$

$$L_k Q_\lambda = \sum_{i=1}^{2m} (\lambda_i - k)\, Q_{\lambda - 2k\epsilon_i},$$

where $\lambda - 2k\epsilon_i = (\lambda_1, \dots, \lambda_i - 2k, \dots, \lambda_{2m})$.

Theorem 3. ([2])
Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$ be a positive integer sequence. Then

$$L_{-k} Q_\alpha = \sum_{i=1}^{\ell} (\alpha_i + k)\, Q_{\alpha+2k\epsilon_i} + \frac{1}{2}\sum_{i=0}^{k-1}(-1)^i (k-i) Q_{\alpha,2k-i,i}, \quad k \ge 1.$$

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4次元Wess-Zumino-Witten模型のソリトン解と共鳴

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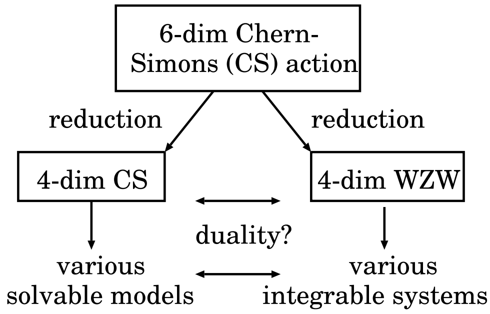

4次元Wess-Zumino-Witten(4dWZW)模型は2次元WZW模型の高次元版であり, 共形場理論としての側面を持つ[13, 9, 15]. 一方、4次元WZW模型の運動方程式はヤンの方程式(反自己双対ヤン・ミルズ(=ASDYM)方程式と等価)であり, ツイスター理論としての側面を持つ[14]. 特にsplit計量(++−−)の場合はこの模型はN=2開弦理論の弦の場の理論を記述し[17], また次元還元からさまざまな可積分方程式を与えることが知られている(Ward予想)[18]. 一方, 2017年頃Costello氏＆Witten氏＆山崎雅人氏は, 可解系のスペクトルパラメータの起源を突き詰めることで, 4次元チャーンサイモンズ理論(4dCS)がさまざまな可解系(スピン系やカイラル模型など)を生み出す親玉であることを見出した[3, 2]. さらに2020年頃にCostello氏の示唆を受けてBittleston氏＆Skinner氏が6次元チャーンサイモンズ理論(6dCS)から4dCSと4dWZWがそれぞれ違う方向へのリダクション過程で得られることを証明した[1]. この(6dCS→4dCS/4dWZW)の枠組みこそが可積分系の統一理論ではなかろうかと多くの研究者が考えており世界的に活発な研究がなされている. またこのようなダブル・ファイブレーションの状況においては4dCSと4dWZWの間に興味深い対応関係が成り立つことが予想される[7]. これは上述の広範囲にわたる可解系と可積分系の間の未知の対応関係を予言しており大変興味深い.

ヤンの方程式(反自己双対ヤン・ミルズ方程式)のソリトン解はこれまでいくつかの例が知られていたが, ダルブー変換により構成されたWronskian解[16]から得られるソリトン解[4]が大変興味深い性質を持つ. まずヤン・ミルズ理論の枠組みで, ゲージ群のランクが2の場合に, 作用密度が実数値でかつ3次元超平面上に局在する(余次元1の)1ソリトンが見出された[5]. 次いでnソリトン解の漸近的振る舞いがKPソリトンと類似した「n個の1ソリトンの非線形重ね合わせ」であることが見出され, 位相のずれ(phase-shift)も計算された[6]. こうして佐藤理論との接点が解のレベルで初めて明らかになったが, さらに4dWZW模型の作用密度についてもこれらの解の振る舞いが明らかにされ, やはりKPソリトンと類似したものであることが分かった[8]. 実数値KPソリトンの分類は児玉裕治氏たちによってpositive Grassmannianの言葉で見事に与えられており[11], ASDYMソリトンの分類への道筋に光明が差している. KPソリトンのソリトングラフの基本要素は「Y字型」ソリトンであり, これは位相のずれ無限大の極限で与えられる共鳴状態であるとも解釈できる.

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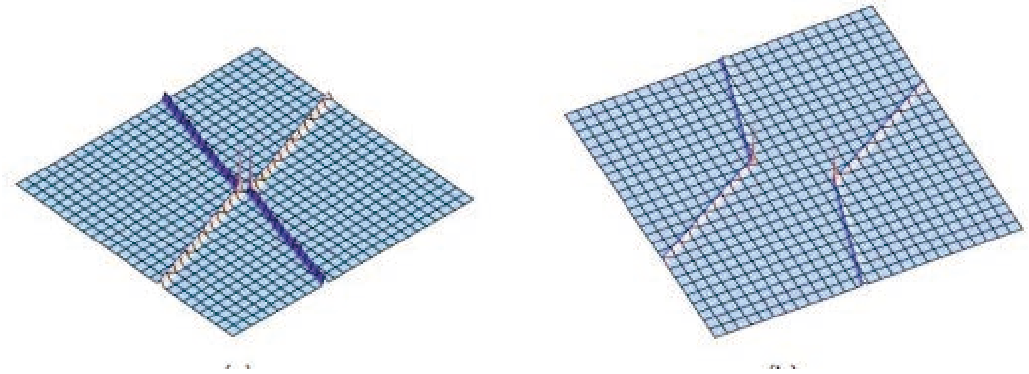
この講演では, 4次元WZW模型のソリトン解を紹介し, 2ソリトン解のLarge phase-shift極限(共鳴極限)を議論する[10]. その結果, KPソリトンと違って, 2つのY字型ソリトンの繋がった共鳴状態ではなく, 2つのV字型ソリトンの繋がった共鳴状態となることが分かった(下図). またこれとは逆符号の場合のLarge phase-shift極限を取ると, mKdV方程式などで見られるDouble Pole共鳴状態が現れることが分かった. これらの共鳴現象はKPソリトンと異なり大変興味深いが, ASDYM方程式から次元還元で得られる方程式で見られる共鳴現象と類似である. なお[10]で議論されるソリトン解はbinary Darboux変換で構成されたGrammian型の解である. コーシー行列によるアプローチ[12]との関係も明らかにされている[10]. 余力があれば, これらの現象のN=2弦理論における解釈や今後の研究方向(4次元Chern-Simons 理論との関連や非可換空間への拡張など) についても議論したい[7].



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Large BKP vs. B-Toda in Lax-Sato form

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sを離散変数，$\boldsymbol{t}=(t_1,t_2,\cdots)$を連続変数の組とする．**大BKP (large BKP)** 階層はJimbo-Miwa[1]，Kac-van de Leur[2] によって導入され[1]，τ 函数$\tau(s,\boldsymbol{t})$に対する双線形方程式

$$\oint z^{s-s'}e^{\xi(\boldsymbol{t}-\boldsymbol{t}',z)}\tau(s-1,\boldsymbol{t}-[z^{-1}])\tau(s'+1,\boldsymbol{t}'+[z^{-1}])\frac{dz}{2\pi iz} + \oint z^{s'-s}e^{\xi(\boldsymbol{t}'-\boldsymbol{t},z)}\tau(s+1,\boldsymbol{t}+[z^{-1}])\tau(s'-1,\boldsymbol{t}'-[z^{-1}])\frac{dz}{2\pi iz} = \frac{1}{2}\left(1-(-1)^{s+s'}\right) \quad (\clubsuit)$$

で定義される．ここで$[z]=(z,z^2/2,\cdots,z^k/k,\cdots)$, $\xi(\boldsymbol{t},z)=t_1z+t_2z^2+\cdots$ という慣用の記法を用いている．Guan ら [4] は波動函数

$$\Psi_+(s,\boldsymbol{t},z)=\frac{\tau(s-1,\boldsymbol{t}-[z^{-1}])}{\tau(s,\boldsymbol{t})}z^se^{\xi(\boldsymbol{t},z)},\quad \Psi_-(s,\boldsymbol{t},z)=\frac{\tau(s+1,\boldsymbol{t}+[z^{-1}])}{\tau(s,\boldsymbol{t})}z^{-s-1}e^{-\xi(\boldsymbol{t},z)}$$

を導入し，着付け作用素$S_\pm=a_0^\pm+a_1^\pm\Lambda^{-1}+a_2^\pm\Lambda^{-2}+\cdots$ $(\Lambda=e^{\partial/\partial s})$ を

$$\Psi_+(s,\boldsymbol{t},z)=S_+z^se^{\xi(\boldsymbol{t},z)},\quad \Psi_-(s,\boldsymbol{t},z)=S_-z^{-s-1}e^{-\xi(\boldsymbol{t},z)}$$

が成り立つように定めて，$(\clubsuit)$を$S_\pm$に対する作用素方程式

$$S_+(\boldsymbol{t})e^{\xi(\boldsymbol{t}-\boldsymbol{t}',\Lambda)}\Lambda^{-1}S_-(\boldsymbol{t}')+S_+(\boldsymbol{t})^*e^{\xi(\boldsymbol{t}'-\boldsymbol{t},\Lambda^{-1})}\Lambda S_-(\boldsymbol{t}')^*=\frac{1}{2}\sum_{n=-\infty}^{\infty}(1-(-1)^n)\Lambda^n \quad (\spadesuit)$$

に書き直した．ここで * は差分作用素の形式的共役を表す．たとえば$(a(s)\Lambda^n)^*=\Lambda^{-n}\cdot a(s)=a(s-n)\Lambda^{-n}$となる．これから$S_\pm$に対する時間発展の方程式（佐藤方程式）やラックス作用素

$$L_+=S_+\Lambda S_+,\quad L_-=S_-^*\Lambda^{-1}S_-^{*-1}$$

に対するラックス方程式が得られる（いずれも具体的な形は省く）．

Guan らはこのラックス-佐藤形式を用いてZabrodin らの**B型戸田階層**（B-Toda hierarchy）[5, 6] との対応関係も論じた．B 型戸田階層は２次元戸田階層の２つのラックス作用素 $L=u_0\Lambda+u_1+u_2\Lambda^{-1}+\cdots$, $\bar{L}=\bar{u}_0\Lambda^{-1}+\bar{u}_1+\bar{u}_2\Lambda+\cdots$ （それぞれL_+,L_-に対応する）に対して

$$L^*(\Lambda-\Lambda^{-1})=(\Lambda-\Lambda^{-1})\bar{L}$$

という束縛条件を課すことによって得られる．その帰結として$L,\bar{L}$の主導係数は**バランス条件** $u_0=\bar{u}_0$を満たす．さらに，この束縛条件を保つために，２次元戸田階層の２系列の時間変数$\boldsymbol{t}=(t_1,t_2,\cdots)$, $\bar{\boldsymbol{t}}=(\bar{t}_1,\bar{t}_2,\cdots)$も反対角線$\boldsymbol{t}+\bar{\boldsymbol{t}}=\boldsymbol{0}$の上に制限される．残された時間変数$t_1,t_2,\cdots$に関するラックス方程式と佐藤方程式は

$$\frac{\partial L}{\partial t_k}=[A_k,L],\quad \frac{\partial \bar{L}}{\partial t_k}=[A_k,\bar{L}],$$
$$\frac{\partial W}{\partial t_k}=A_kW-W\Lambda^k,\quad \frac{\partial \bar{W}}{\partial t_k}=A_k\bar{W}+\bar{W}\Lambda^{-k}$$

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[1] Kac-van de Leur[2] はそれを**フェルミオン的BKP階層**と呼んだ．大BKP階層という名称はそれまで知られていたBKP階層（**小BKP階層**）と区別するためにOrlov-Shiota-Takasaki[3] によって導入された．

という形になる．ここで A_k は

$$A_k^*(\Lambda - \Lambda^{-1}) = -(\Lambda - \Lambda^{-1})A_k$$

という条件を満たす有限階差分作用素（佐藤方程式自体から定まる）である．$W, \bar{W}$ は $W = w_0 + w_1\Lambda^{-1} + w_2\Lambda^{-2} + \cdots$, $\bar{W} = \bar{w}_0 + \bar{w}_1\Lambda + \bar{w}_2\Lambda^2 + \cdots$ という形の差分作用素で，着付け関係 $L = W\Lambda W^{-1}$, $\bar{L} = \bar{W}\Lambda^{-1}\bar{W}^{-1}$ が成り立つ．

このB型戸田階層の定式化をGuanらの大BKP階層の記述と見比べれば，以下のことがわかる．1は2つの論文[4, 5]から読み取れる．2,3が本講演の主結果である．

1. $W, \bar{W}$ は $S_\pm$ と
$$S_+ = W(1-\Lambda^{-2})^{-1/2}, \quad S_- = (1-\Lambda^{-2})^{-1/2}\bar{W}^*$$
という関係で結ばれていて
$$W^*(\Lambda - \Lambda^{-1})\bar{W} = \Lambda - \Lambda^{-1}$$
という束縛条件を満たす．その帰結として**バランス条件** $w_0(s)\bar{w}_0(s-1) = 1$ が成り立つ．こうして佐藤方程式においても大BKP階層とB型戸田階層は対応している．

2. $U = W^{-1}\bar{W}$ は微分方程式
$$\frac{\partial U}{\partial t_k} = \Lambda^k U + U\Lambda^k$$
と代数的関係式
$$U^*(\Lambda - \Lambda^{-1}) = (\Lambda - \Lambda^{-1})U \qquad (\heartsuit)$$
を満たす．特に $\boldsymbol{t} = \boldsymbol{0}$ における初期値 $U_0 = U(\boldsymbol{0})$ はこの代数的関係式を満たし，U は
$$U = \exp\left(\sum_{k=1}^\infty t_k\Lambda^k\right) U_0 \exp\left(\sum_{k=1}^\infty t_k\Lambda^{-k}\right) \qquad (\diamondsuit)$$
と表せる．

3. 逆に，$U_0^*(\Lambda - \Lambda^{-1}) = (\Lambda - \Lambda^{-1})U_0$ を満たす差分作用素 U_0 に対して U を $(\diamondsuit)$ によって定めれば，U は $(\heartsuit)$ を満たす．この U を
$$U = W^{-1}\bar{W}$$
というように因子分解すれば（ゲージ変換 $W \to gW$, $\bar{W} \to g\bar{W}$ を利用して，バランス条件を満たすように $W, \bar{W}$ を選び直せる），$W, \bar{W}$ は佐藤方程式を満たし，$L = W\Lambda W^{-1}$, $\bar{L} = \bar{W}\Lambda^{-1}\bar{W}^{-1}$ はB型戸田階層の解を与える．

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KP Solitons and the Schottky Uniformization

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Abstract

Real and regular soliton solutions of the KP hierarchy have been classified in terms of the totally nonnegative (TNN) Grassmannians. These solitons are referred to as KP solitons, and they are expressed as singular (tropical) limits of shifted Riemann theta functions. In this talk, for each element of the TNN Grassmannian, we construct a Schottky group, which uniformizes the Riemann surface associated with a real finite-gap solution. Then we show that the KP solitons are obtained by degenerating these finite-gap solutions.

The talk is based on a collaborative work with Takashi Ichikawa (Saga University) [11].

1. Introduction

It is known that solutions of the KP equation can be constructed from *any* algebraic curves (Riemann surfaces) [19]. A solution from a smooth curve is a *quasi-periodic* solution, and some soliton solutions can be constructed by rational (tropical) limits of the curve with only ordinary double points, i.e. a singular Riemann surface with nodal singularities (see e.g. [21, 26, 2]). In particular, the cases corresponding to the KdV and nonlinear Schödinger equations are well-studied, in which the algebraic curves are given by the hyperelliptic curves (see e.g. [2, 21]). Recently, there are several papers dealing with some non-hyperelliptic cases, e.g. so-called (n, s)-curves, where the authors construct the Klein σ-functions over these curves (see e.g. [3, 18, 20, 22]). It seems, however, that almost no result has been reported for the cases with more general algebraic curves. Because of the difficulty in finding a canonical homological basis for the general algebraic curves, it may be quite complicated to compute explicitly a rational limit of these curves and the corresponding Riemann theta functions (see [22]). On the other hand, a large number of *real* and *regular* soliton solutions of the KP hierarchy, referred to as *KP solitons*, has been classified in terms of totally nonnegative (TNN) Grassmannian $\mathrm{Gr}(N, M)_{\geq 0}$ (see e.g. [16, 15, 13]). We also mention that recently, there are some progress on the study concerned with the connections between the algebraic-geometric solutions and these soliton solutions [1, 23, 24, 17].

In this talk, we first give a brief review of the KP solitons with combinatorial aspects of the TNN Grassmannians. In particular, we describe some details of the so-called Ⅎ-*diagram*, introduced by Postnikov [25], which provides a parametrization of the KP solitons. In [14], we identify singular Riemann surfaces for the KP solitons, and introduce the *M-theta function* defined on the singular Riemann surface. The M-theta function is obtained by singular (rational) limit of the Riemann theta function, and it gives the τ-function of the KP soliton.

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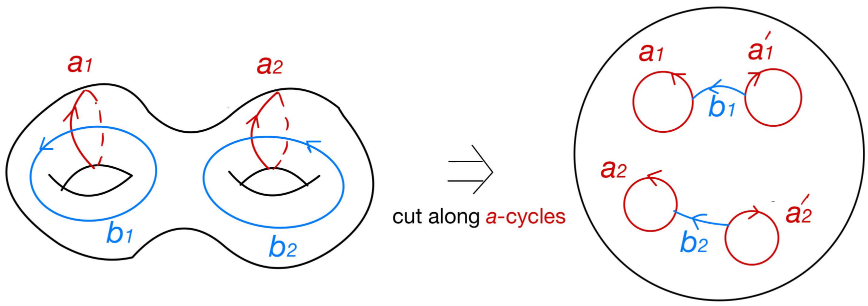
Then using the Schottky uniformization theory [9], we construct real smooth Riemann surfaces associated with finite gap solutions of the KP equation. In particular, we show that the Ⅎ-diagram in the TNN Grassmanian theory is quite useful for the construction. More precisely, the Ⅎ-diagram can provide the information about a canonical homological basis for the smooth Riemann surface.

2. The compact Riemann surface and the theta function

Let $\mathcal{R}_g$ be a smooth compact Riemann surface of genus g. Let $H_1(\mathcal{R}_g, \mathbb{Z})$ be the homology group of $\mathcal{R}_g$, and a set $\{a_1, \ldots, a_g, b_1, \ldots, b_g\}$ be a canonical basis in $H_1(\mathcal{R}_g, \mathbb{Z})$, that is, we have the intersection products,

$$a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad a_j \circ b_k = \delta_{j,k}.$$

It is well-known that any compact Riemann surface of genus g is homeomorphic to a sphere with g handles (see e.g. [5]). The left panel of the figure below shows a Riemann surface of genus 2. Cutting the Riemann surface along the a-cycles, we



obtain the manifold $\mathbb{CP}^1$ with $2g$ holes, as shown in the right panel of the figure. This implies that the Riemann surface can be obtained by identifying each pair of a- and a'-cycles. The identification can be expressed by a Schottky group [6] as shown in Section 6, which is the main theme in the present note.

Given a set of canonical basis of $H_1(\mathcal{R}_g, \mathbb{Z})$, we have the holomorphic differentials $\{\omega_j : j = 1, \ldots, g\}$ normalized by the conditions,

$$\oint_{a_j} \omega_k = \delta_{j,k}, \qquad (1 \le j, k \le g).$$

The integrals over the b-cycles given by

$$\Omega_{j,k} := \oint_{b_j} \omega_k, \qquad (1 \le j < k \le g) \tag{2.1}$$

define the $g \times g$ period matrix $\Omega = (\Omega_{j,k})$, which is symmetric and $\mathrm{Im}(\Omega) > 0$. Then the Riemann theta function associated with $\mathcal{R}_g$ is defined by

$$\vartheta_g(\mathbf{z}; \Omega) := \sum_{\mathbf{m} \in \mathbb{Z}^g} \exp 2\pi i \left(\frac{1}{2} \mathbf{m}^T \Omega \mathbf{m} + \mathbf{m}^T \mathbf{z} \right), \tag{2.2}$$

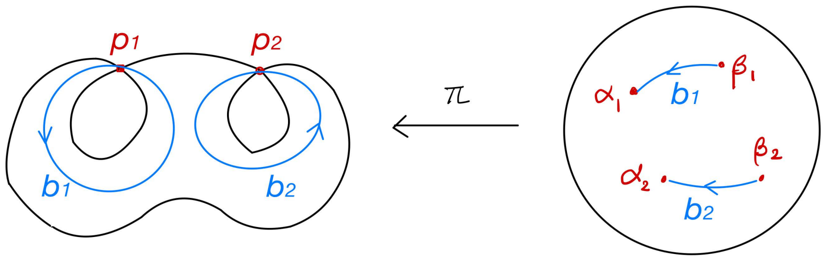
for $\mathbf{z} \in \mathbb{C}^g$, and $\mathbf{m}^T$ is the transpose of the column vector $\mathbf{m} \in \mathbb{Z}^g$.

2.1. The Riemann theta function on a singular curve

In [21] (Chapter 5, p.3.243), Mumford considered the theta function on singular curve. Let $\widetilde{\mathcal{R}}_g$ be a singular Riemann surface of (arithmetic) genus g corresponding to the curve $\mathcal{C}$, and let S be the set of singular points, $S = \{p_1, \dots, p_g\} \subset \widetilde{\mathcal{R}}_g$. Assume that the singularities of $\widetilde{\mathcal{R}}_g$ are only ordinary double points $p_1, \dots, p_g$ and that $\widetilde{\mathcal{R}}_g$ has normalization

$$\pi : \mathbb{CP}^1 \longrightarrow \widetilde{\mathcal{R}}_g \quad \text{with} \quad \pi^{-1}(p_i) = \{\alpha_i, \beta_i\} \tag{2.3}$$

That is, $\widetilde{\mathcal{R}}_g$ is just $\mathbb{CP}^1$ with g pairs of points $\{\alpha_i, \beta_i\}$ identified. Figure below shows the case with $g = 2$. The singular Riemann surface $\widetilde{\mathcal{R}}_g$ is obtained by



pinching all a-cycles as shown in the figure.

By pinching a-cycles, the holomorphic differentials $\{\omega_k : k = 1, \dots, g\}$ take the limits [12, 10] (see also Section 6.1),

$$\omega_k \quad \longrightarrow \quad \widetilde{\omega}_k = \frac{dz}{2\pi i}\left(\frac{1}{z-\alpha_k} - \frac{1}{z-\beta_k}\right). \tag{2.4}$$

Then the period matrix in (2.1) becomes

$$\Omega_{j,k} \quad \longrightarrow \quad \widetilde{\Omega}_{j,k} := \int_{\beta_j}^{\alpha_j} \widetilde{\omega}_k = \frac{1}{2\pi i} \ln C_{j,k} \qquad \text{mod}(\mathbb{Z}), \tag{2.5}$$

where $C_{j,k}$ is given by the cross-ratio $[\alpha_j, \beta_j; \alpha_k, \beta_k]$,

$$C_{j,k} = [\alpha_j, \beta_j; \alpha_k, \beta_k] := \frac{(\alpha_j - \alpha_k)(\beta_j - \beta_k)}{(\alpha_j - \beta_k)(\beta_j - \alpha_k)}. \tag{2.6}$$

Note in particular that the diagonal parts of the period matrix Ω has the limits

$$\text{Im}\ \Omega_{i,i} \longrightarrow \infty \quad \text{for} \quad 1 \le i \le g, \tag{2.7}$$

Then the limit of the ϑ-function (2.2) is just 1, which corresponds to the choice $\mathbf{m}^T = (0, \dots, 0)$. To obtain a nontrivial example, we consider the shifts

$$z_i \longrightarrow z_i - \frac{1}{2}\Omega_{i,i}, \quad \text{for} \quad i = 1, \dots, g,$$

which then gives the Riemann theta function with shifted variable $\mathbf{z} \in \mathbb{C}^g$,

$$\vartheta_g(\mathbf{z};\Omega) = \sum_{\mathbf{m}\in\mathbb{Z}^g} \exp 2\pi i \left(\frac{1}{2}\sum_{i=1}^{g} m_i(m_i-1)\Omega_{i,i} + \sum_{i<j} m_i m_j \Omega_{i,j} + \sum_{i=1}^{g} m_i z_i \right). \tag{2.8}$$

Then the limit $\Omega_{j,j} \to +i\infty$ for all $j = 1,\dots,g$ leads to

$$\vartheta_g(\mathbf{z};\Omega) \;\longrightarrow\; \tilde{\vartheta}_g(\mathbf{z};\tilde{\Omega}) := \sum_{\mathbf{m}\in\{0,1\}^g} \exp 2\pi i \left(\sum_{j<k} m_j m_k \tilde{\Omega}_{j,k} + \sum_{n=1}^{g} m_n z_n \right) \tag{2.9}$$

$$= 1 + \sum_{n=1}^{g} e^{2\pi i z_n} + \sum_{j<k} C_{j,k} e^{2\pi i (z_j+z_k)} + \ldots + \left(\prod_{j<k} C_{j,k} \right) e^{2\pi i \sum_{n=1}^{g} z_n},$$

Note that the infinite sum of exponential terms in the ϑ-function (2.8) becomes a *finite* sum of 2^g exponential terms with $m_i \in \{0,1\}$, if all $C_{j,k} \neq 0$ for $j<k$. The function $\tilde{\vartheta}_g$ is referred to as the M-theta function [14].

Remark 2.1. *When all the pairs $\{\alpha_k, \beta_k\}$ are real and $\alpha_k < \beta_k$ w.l.o.g., one should note that the cross ratio $C_{j,k}$ in* (2.6) *takes the signs depending on the orders of the pairs, i.e.*

(i) *if $\alpha_j < \beta_j < \alpha_k < \beta_k$ or $\alpha_j < \alpha_k < \beta_k < \beta_j$, then $C_{j,k} > 0$,*

(ii) *if $\alpha_j < \alpha_k < \beta_j < \beta_k$, then $C_{j,k} < 0$, and*

(iii) *if $\alpha_j = \alpha_k$ or/and $\beta_j = \beta_k$, then $C_{j,k} = 0$.*

The case (ii) *will be important when we discuss the regularity of the soliton solutions (see also* [8]*). Also note that the case* (iii) *implies that the off-diagonal element $\widetilde{\Omega}_{j,k}$ takes $+i\infty$, in addition to the diagonal elements in the singular limit* (2.7)*.*

3. The KP equation

In this section, we give a brief summary of the KP solitons for the purpose of the present paper (see e.g. [13] for the details). The KP equation is a nonlinear partial differential equation in the form,

$$\partial_x(-4\partial_t u + 6u\partial_x u + \partial_x^3 u) + 3\partial_y^2 u = 0, \tag{3.1}$$

where $\partial_z^k := \frac{\partial^k}{\partial z^k}$ for $z = x, y, t$. The solution of the KP equation is given in the following form,

$$u(x,y,t) = 2\partial_x^2 \ln \tau(x,y,t), \tag{3.2}$$

where $\tau(x,y,t)$ is called the τ-function of the KP equation.

3.1. Soliton solutions

The soliton solutions are constructed as follows: Let $\{f_i(x,y,t) : 1 \le i \le N\}$ be a set of linearly independent functions $f_i(x,y,t)$ satisfying the following system of linear equations,

$$\partial_y f_i = \partial_x^2 f_i, \quad \text{and} \quad \partial_t f_i = \partial_x^3 f_i \qquad i = 1, \dots, N. \tag{3.3}$$

The Wronskian $\mathrm{Wr}(f_1, \dots, f_N)$ with respect to the x-variable gives a τ-function, that is, the function $u(x,y,t)$ in (3.2) is a solution of the KP equation,

$$\tau(x,y,t) = \mathrm{Wr}(f_1, f_2, \dots, f_N). \tag{3.4}$$

(See, e.g. [13] for the details.)

As a fundamental set of the solutions of (3.3), we take the exponential functions $E_j(x,y,t)$ for $j = 1, \dots, M$ $(M > N)$, i.e.

$$E_j(x,y,t) = e^{\xi_j(x,y,t)} \quad \text{with} \quad \xi_j(x,y,t) := \kappa_j x + \kappa_j^2 y + \kappa_j^3 t. \tag{3.5}$$

where κ_j's are arbitrary real constants. In this paper, we consider the regular soliton solutions, for which we assume the ordering

$$\kappa_1 \ < \ \kappa_2 \ < \ \cdots \ < \ \kappa_M. \tag{3.6}$$

For the soliton solutions, we consider $f_i(x,y,t)$ as a linear combination of the exponential solutions,

$$f_i(x,y,t) = \sum_{j=1}^{M} a_{i,j} E_j(x,y,t) \qquad \text{for} \qquad i = 1, \dots, N. \tag{3.7}$$

where $A := (a_{i,j})$ is an $N \times M$ constant matrix of full rank, $\mathrm{rank}(A) = N$. Then the τ-function (3.4) is expressed by

$$\tau(x,y,t) = |AE(x,y,t)^T|, \tag{3.8}$$

where $E(x,y,t)^T$ is the transpose of the $N \times M$ matrix $E(x,y,t)$ defined by

$$E(x,y,t) = \begin{pmatrix} E_1 & E_2 & \cdots & E_M \\ \kappa_1 E_1 & \kappa_2 E_2 & \cdots & \kappa_M E_M \\ \vdots & \vdots & \ddots & \vdots \\ \kappa_1^{N-1} E_1 & \kappa_2^{N-1} E_2 & \cdots & \kappa_M^{N-1} E_M \end{pmatrix}. \tag{3.9}$$

Note here that the set of exponential functions $\{E_1, \dots, E_M\}$ gives a basis of M-dimensional space of the null space of the operator $\prod_{i=1}^{M}(\partial_x - \kappa_i)$, and we call it a *basia* of the KP soliton. Then the set of functions $\{f_1, \dots, f_N\}$ represents an N-dimensional subspace of M-dimensional space spanned by the exponential functions. This leads naturally to the structure of a finite real Grassmannian $\mathrm{Gr}(N,M)$, the set of N-dimensional subspaces in $\mathbb{R}^M$. Then the $N \times M$ matrix A of full rank can be identified as a point of $\mathrm{Gr}(N,M)$, and throughout the paper we assume A to be in the reduced row echelon form (RREF).

Definition 3.1. *An $N \times M$ matrix A in RREF is irreducible, if*

(a) *in each row, there is at least one nonzero element besides the pivot, and*

(b) *there is no zero column.*

This implies that the first pivot is located at $(1, 1)$ entry, and the last pivot should be at (N, i_N) with $N \le i_N < M$.

The τ-function in (3.8) can be expressed as the following formula using the Binet-Cauchy lemma (see e.g. [13]),

$$\tau(x, y, t) = \sum_{I \in \binom{[M]}{N}} \Delta_I(A) E_I(x, y, t), \tag{3.10}$$

where $I = \{i_1 < i_2 < \cdots < i_N\}$ is an N element subset in $[M] := \{1, 2, \ldots, M\}$, $\Delta_I(A)$ is the $N \times N$ minor with the column vectors indexed by $I = \{i_1, \ldots, i_N\}$, and $E_I(x, y, t)$ is the $N \times N$ determinant of the same set of the columns in (3.9), which is given by

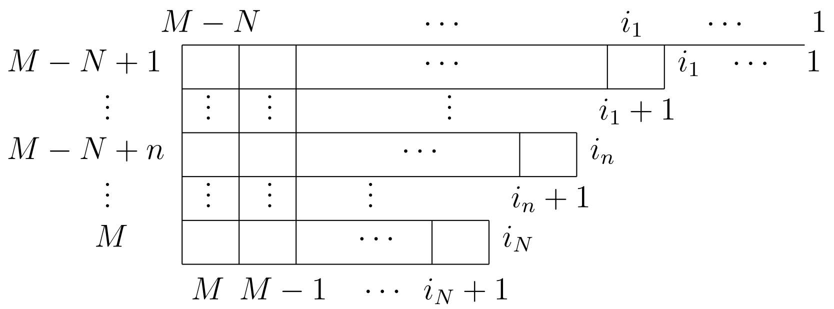
$$E_I = \prod_{k<l} (\kappa_{i_l} - \kappa_{i_k}) \, E_{i_1} \cdots E_{i_N} = \prod_{k<l} (\kappa_{i_l} - \kappa_{i_k}) \exp\left(\xi_{i_1} + \cdots + \xi_{i_N}\right). \tag{3.11}$$

The minor $\Delta_I(A)$ is also called the Plücker coordinate, and the τ-function represents a point of $\mathrm{Gr}(N, M)$ in the sense of the Plücker embedding, $\mathrm{Gr}(N, M) \hookrightarrow \mathbb{P}(\wedge^N \mathbb{C}^M) : A \mapsto \{\Delta_I(A) : I \in \binom{[M]}{N}\}$. It is then obvious that if all the minors of A are nonnegative, the τ-function (3.10) is sign-definite, i.e. the solution u in (3.2) is *regular*. The $\mathrm{Gr}(M, N)$ consisting of these elements is called the totally nonnegative (TNN) Grassmannian, denoted by $\mathrm{Gr}(N, M)_{\ge 0}$. Then the following theorem for the *necessary* condition of the regularity was proven in [16].

Theorem 3.2. *The soliton solution generated by the τ-function* (3.10) *is regular if and only if the matrix A is in* $\mathrm{Gr(N, M)}_{\ge 0}$.

4. Combinatorics for the TNN Grassmannians

We here provide a brief summary of combinatorial description of the TNN Grassmannian $\mathrm{Gr}(N, M)_{\ge 0}$ (see also [13] for the details). Each element $A \in \mathrm{Gr}(N, M)$ is expressed as an $N \times M$ matrix in the reduced row echelon form. Let $\{i_1, \ldots, i_N\}$ be the pivot set of the matrix A. Then the Young diagram corresponding to the pivot set is obtained as follows: Consider a lattice path starting from the top right corner and ending at the bottom left corner with the label $\{1, \ldots, M\}$, so that the pivot indices appear at the vertical paths as shown in the diagram below.



We recall that the partitions λ are in bijection with N-element subset $I \subset [M]$, i.e. we have $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$ with

$$\lambda_k = M - N - (i_k - k) \qquad \text{for} \quad k = 1, \ldots, N.$$

The irreducible element $A \in \mathrm{Gr}(N, M)_{\geq 0}$ defines the *irreducible* Young diagram, which has $\lambda_1 = M - N$ and $\lambda_N \geq 1$.

4.1. The Le-diagram

In [25], Postnikov introduced the ⅃-diagram (called *Le*-diagram), which gives a unique parametrization of the element $A \in \mathrm{Gr}(N, M)_{\geq 0}$.

Definition 4.1. *A ⅃-diagram is a decorated Young diagram with ◯ in some boxes, which satisfies the property (called ⅃-property): If there is ◯, then all the boxes either to its left or above it are all ◯. That is, there is no such ◯, which has an empty box to its left and an empty box above it. We also say that a ⅃-diagram is irreducible, if each column and row has at least one empty box (i.e no zero column or/and no zero row). See the left diagram in Example 4.3 below.*

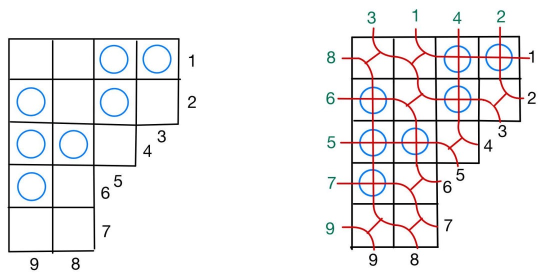
Then Postnikov proved the following theorem.

Theorem 4.2. *There is a bijection between the set of irreducible ⅃-diagram and the set of derangements of the symmetric group S_M.*

Here the derangement associated to the ⅃-diagram can be found by constructing a *pipedream* on the diagram as follows (see [13] for the details): Starting from a ⅃-diagram, we replace a blank box with a box containing elbow-pipes connected by a bridge and replace a box with ◯ by a box containing crossing pipes as shown below. Then we label the southeast (input) boundary of the ⅃-diagram from 1 to

M starting from the top corner to the bottom corner of the boundary. We place a pipe with the index of the input edge from the southeast (output) boundary to the northwest boundary, and then label each northwest edge according to the index of the pipe. Then the derangement σ with a pair (i, j) in $\sigma(i) = j$ can be found on the opposite sides of the boundary.

Example 4.3. *Below shows a ⅃-diagram and its pipedream, The derangement*



corresponding to the pipedream is $(8, 6, 2, 5, 4, 7, 9, 1, 3)$ *in one-line notation.*

One can also show the following proposition from the ⅃-diagram.

Proposition 4.1. *Given an irreducible ⅃-diagram, the zero entries of* $A \in Gr(N, M)_{\geq 0}$ *can be determined as follows: Consider a box at* (i_k, j) *with* ◯ *whose south-east conner is a point of the boundary of the diagram, and recall the ⅃-property. We have two cases as shown in the figure below.*

(a) *The* k*-th row, say* $A_{k,\bullet}$*, of the matrix* A *has the structure,*

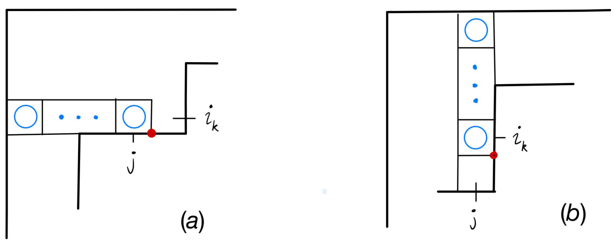
$$A_{k,\bullet} = (\dots, 0, 1, \dots, *, 0, 0, \dots, 0),$$

that is, the pivot "1" is at (k, i_k) *and the nonzero element marked by "*" is at* $(k, j-1)$*. The entries* $A_{k,l}$ *for* $j \leq l \leq M$ *are all zero.*

(b) *The* j*-th column, say* $A_{\bullet,j}$*, of the matrix* A *has the structure,*

$$(A_{\bullet,j})^T = (0, 0, \dots, 0, *, \dots),$$

that is, the entries $A_{l,j} = 0$ *for* $1 \leq l \leq k$ *are all zero.*



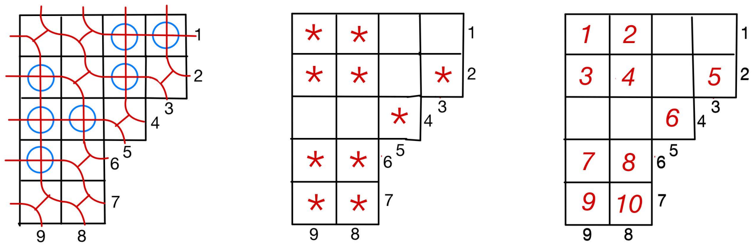
Proof. Using Theorem 5.6 in [16] about the vanishing minors, one can show

(a) the minor $\Delta_{i_1,\dots,i_{k-1},j,i_{k+1},\dots,i_N}(A) = 0$, and

(b) the minor $\Delta_{i_1,\dots,i_{k-1},i_{k+1},\dots,i_l,j,i_{l+1},\dots,i_N}(A) = 0$,

which imply the equations in the proposition. Note that there is a case $j > i_{k+1}$ in (a). This can can be also proven in the same way. □

Example 4.4. *Consider the example 4.3. The middle diagram in the figure below shows the nonzero entries other than pivots in the matrix* A*, e.g.* $A_{2,8} \neq 0$ *and* $A_{3,5} \neq 0$*. Each empty box gives zero entry of* A*, e.g.* $A_{1,5} = A_{3,8} = 0$*.*



Each star in the middle diagram implies that there is a path $[i, j]$ *through the pipedream from the pivot index* i *at the east boundary to the non-pivot index* j *at the south boundary of the ⅃-diagram.*

For $A \in \mathrm{Gr}(N,M)_{\geq 0}$, we define the matroid,

$$\mathcal{M}(A) = \left\{ I \in \binom{[M]}{N} : \Delta_I(A) \right\}. \tag{4.1}$$

Let I_0 be the lexicographically minimum element of $\mathcal{M}(A)$. Then we have the decomposition,

$$\mathcal{M}(A) = \bigcup_{n=0}^{N} \mathcal{M}_n(A), \tag{4.2}$$

where

$$\mathcal{M}_n(A) := \{ J \in \mathcal{M}(A) : |J \cap I_0| = N - n \}.$$

Note that $\mathcal{M}_0(A) = I_0$. We also define $P_1(A)$ as the set of pairs $[i,j]$,

$$P_1(A) := \{ [i,j] : i \in I_0 \setminus J, j \in J \setminus I_0 \text{ for } J \in \mathcal{M}_1(A) \} \tag{4.3}$$

This implies that $P_1(A)$ is identified as the set of nonzero entries in A besides the pivots, that is, $[i_k, j_l] \in P_1(A)$ represents

(a) $i_k \in I_0 \setminus J$ is the k-th pivot of A, i.e. $A_{k,i_k} = 1$,

(b) $j_l \in J \setminus I_0$ is the nonzero element A_{k,j_l} in the k-th row.

One can define the order in $P_1(A)$: Let ℓ be a bijection satisfying the following order,

(1) $\ell([i,k]) < \ell([i,l])$, if $k > l$,

(2) $\ell([i,\bullet]) < \ell([j,\bullet])$, if $i < j$.

Then the elements of $P_1(A)$ can be uniquely numbered from 1 to $|P_1(A)|$, i.e.

$$1 \;\leq\; \ell([i,j]) \;\leq\; g, \quad \text{for } \; [i,j] \in P_1(A), \tag{4.4}$$

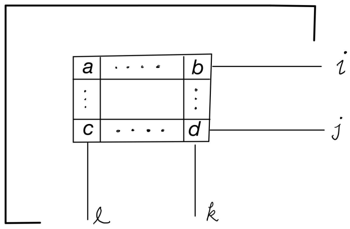
where $g = |P_1(A)|$. Note that (4.4) gives the ordering of the singular points in the nomalization (2.3),

$$\pi^{-1}(p_l) = \{\kappa_i, \kappa_j\} \quad \text{for} \quad l = \ell([i,j]), \text{ and } 1 \leq l \leq g. \tag{4.5}$$

As will be shown in the next section, the number g gives the genus of the Riemann surface associated with the KP soliton. We remark that the ordering in (4.5) can be obtained from the Ꞁ-diagram as shown in the right diagram in Example 4.4, and we call the diagram *O*Ꞁ*-diagram*.

From the *O*Ꞁ-diagram, we can also show the following proposition on the sign of the coefficient $C_{p,q}$.

Proposition 4.2. *In the O*Ꞁ*-diagram, consider a rectangular section whose conner boxes are marked a, b, c and d with $a < b < c < d$ as shown in the figure below. We also assign a pair of parameters (κ_p, κ_q) to each box according to the boundary indices of the* Ꞁ*-diagram. Then we have that*



$$a = (\kappa_i, \kappa_l)$$
$$b = (\kappa_i, \kappa_k)$$
$$c = (\kappa_j, \kappa_l)$$
$$d = (\kappa_j, \kappa_k)$$

(i) $C_{a,b} = C_{a,c} = C_{cd} = C_{b,d} = 0$, *and* $C_{a,d} > 0, C_{b,c} < 0$,

(ii) *if one of the conner boxes is empty (no numbered) or the box* $\boxed{d}$ *is outside of the O⅃-diagram, then either* $C_{a,d} > 0$ *or* $C_{b,c} > 0$.

Proof. Note that in the ⅃-diagram, the indices $\{i, j\}$ are pivots, and $\{k, l\}$ are non-pivots. Also we have $\kappa_i < \kappa_j < \kappa_k < \kappa_l$. Then the proof is just the computation of the coefficients given by the cross ratio (2.6). For example, the coefficient $C_{a,d}$ is calculated as

$$C_{a,b} = \frac{(\kappa_i - \kappa_j)(\kappa_l - \kappa_k)}{(\kappa_i - \kappa_k)(\kappa_j - \kappa_l)} > 0$$

It is also easy to show that for the case where $\boxed{d}$ is outside the diagram, we have $C_{b,c} > 0$ (in this case note that $\kappa_i < \kappa_k < \kappa_j < \kappa_l$). □

Example 4.5. *Consider Example 4.4. The following six coefficients are only negative*

$$C_{2,3},\ C_{2,7},\ C_{2,9},\ C_{4,7},\ C_{4,9},\ C_{8,9}\ <\ 0.$$

All other coefficients for $1 \le p < q \le 10$ *are* $C_{p,q} \ge 0$.

5. The τ-function as the M-theta function

The τ-function (3.10) can be expressed as

$$\begin{aligned}\tau(x, y, t) &= \sum_{n=0}^{N} \sum_{J \in \mathcal{M}_n(A)} \Delta_J(A) E_J(x, y, t) \qquad (5.1)\\ &= \Delta_{I_0}(A) E_{I_0}(x, y, t) \left(1 + \sum_{n=1}^{N} \sum_{J \in \mathcal{M}_n(A)} \frac{\Delta_J(A) E_J}{\Delta_{I_0}(A) E_{I_0}} \right)\end{aligned}$$

Since the solution is given by the second derivative of $\ln \tau$, one can take the τ-function in the following form,

$$\tau(x, y, t) = 1 + \sum_{n=1}^{N} \sum_{J \in \mathcal{M}_n(A)} \Delta_J(A) \frac{E_J(x, y, t)}{E_{I_0}(x, y, t)}. \qquad (5.2)$$

where we have taken $\Delta_{I_0}(A) = 1$ for the pivot set I_0.

Then the following theorem is proven in [14].

Theorem 5.1. *Given irreducible $A \in Gr(N,M)_{\geq 0}$, the τ-function* (5.2) *is the M-theta function* (2.9)*, i.e.*

$$\tau(x,y,t) = \vartheta_g(\mathbf{z};\widetilde{\Omega}) = \sum_{m\in\{0,1\}}^{g} \exp 2\pi i \left(\sum_{i<j} m_i m_j \tilde{\Omega}_{i,j} + \sum_{j=1}^{g} m_j z_j \right)$$
$$= 1 + \sum_{p=1}^{g} e^{\tilde{\phi}_p} + \sum_{p<q} C_{p,q} e^{\tilde{\phi}_p + \tilde{\phi}_q} + \cdots + \left(\prod_{p<q} C_{p,q} \right) e^{\sum_{l=1}^{g} \tilde{\phi}_l},$$

where $g = |P_1(A)|$ and $2\pi i z_p = \tilde{\phi}_p(x,y,t) = \phi_p(x,y,t) + \phi_p^0$, and for $p = \ell([i_k, j_l^{(k)}])$ with the ordering ℓ in $P_1(A)$,

$$\phi_p = \xi_{j_m^{(k)}} - \xi_{i_k} = (\kappa_{j_m^{(k)}} - \kappa_{i_k})x + (\kappa^2_{j_m^{(k)}} - \kappa^2_{i_k})y + (\kappa^3_{j_m^{(k)}} - \kappa^3_{i_k})t,$$
$$e^{\phi_p^0} = a_{k,j_m^{(k)}} \frac{\prod_{l\neq k}(\kappa_{i_l} - \kappa_{j_m^{(k)}})}{\prod_{l\neq k}(\kappa_{i_l} - \kappa_{i_k})},$$
$$C_{p,q} = \exp\left(2\pi i\, \tilde{\Omega}_{p,q}\right) = \frac{(\kappa_{i_k} - \kappa_{i_l})(\kappa_{j_m^{(k)}} - \kappa_{j_n^{(l)}})}{(\kappa_{i_k} - \kappa_{j_n^{(l)}})(\kappa_{j_m^{(k)}} - \kappa_{i_l})}.$$

Here $q = \ell([i_l, j_n^{(l)}])$, and $a_{k,j_m^{(k)}}$ is the entry in A corresponding to the element $[i_k, j_m^{(k)}] \in P_1(A)$.

As shown in [8], the sign of $a_{k,j_m^{(k)}}$ is determined by the positivity of $e^{\phi_p^0}$, that is, it is the sign of the product in the equation.

5.1. Example

Consider the $O\perp$-diagram $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$. This implies $g = 4$, and the number in each box of the diagram is assigned by $l = \ell([i,j])$ for $[i,j] \in P_1(A)$ with $A \in \mathrm{Gr}(2,4)_{\geq 0}$, i.e.

$$1 = \ell([1,4]), \quad 2 = \ell([1,3]), \quad 3 = \ell([2,4]), \quad 4 = \ell([2,3]). \tag{5.3}$$

In terms of the normalization (2.3), this ordering means $\pi^{-1}(p_l) = \{\alpha_l, \beta_l\}$ for $l = 1, \ldots, 4$, e.g., $\pi^{-1}(p_2) = \{\kappa_1, \kappa_3\}$ (see (4.5)). Then the coefficients $C_{j,k}$ in (2.6) are calculated as $C_{1,2} = C_{1,3} = C_{2,4} = C_{3,4} = 0$, and

$$C_{1,4} = \frac{(\kappa_1 - \kappa_2)(\kappa_4 - \kappa_3)}{(\kappa_1 - \kappa_3)(\kappa_4 - \kappa_2)} > 0, \qquad C_{2,3} = \frac{(\kappa_1 - \kappa_2)(\kappa_3 - \kappa_4)}{(\kappa_1 - \kappa_4)(\kappa_3 - \kappa_2)} < 0.$$

The matrix $A \in \mathrm{Gr}(2,4)_{\geq 0}$ corresponding to the diagram is given by

$$A = \begin{pmatrix} 1 & 0 & a_{1,3} & a_{1,4} \\ 0 & 1 & a_{2,3} & a_{2,4} \end{pmatrix}.$$

The signs of the entries $a_{i,j}$ are determined by the positivity of $\exp \phi_l^0$, i.e.

$$e^{\phi_1^0} = a_{1,4} \frac{\kappa_2 - \kappa_4}{\kappa_2 - \kappa_1} > 0, \qquad e^{\phi_2^0} = a_{1,3} \frac{\kappa_2 - \kappa_3}{\kappa_2 - \kappa_1} > 0,$$
$$e^{\phi_3^0} = a_{2,4} \frac{\kappa_1 - \kappa_4}{\kappa_1 - \kappa_2} > 0, \qquad e^{\phi_4^0} = a_{2,3} \frac{\kappa_1 - \kappa_3}{\kappa_1 - \kappa_2} > 0,$$

that is, using $\kappa_1 < \kappa_2 < \kappa_3 < \kappa_4$, we have $a_{1,4} < 0, a_{1,3} < 0, a_{2,4} > 0$ and $a_{2,3} > 0$. Notice here that these signs are *not* enough for the total nonnegativity of A (the additional condition is determined by the regularity of the solution [16], see below).

Then the M-theta function (i.e. the τ-function) in Theorem 5.1 is given by

$$\tau = 1 + e^{\widetilde{\phi}_1} + e^{\widetilde{\phi}_2} + e^{\widetilde{\phi}_3} + e^{\widetilde{\phi}_4} + C_{1,4}e^{\widetilde{\phi}_1+\widetilde{\phi}_4} + C_{2,3}e^{\widetilde{\phi}_2+\widetilde{\phi}_3}, \tag{5.4}$$

where the exponents are given by $\widetilde{\phi}_l = \phi_l + \phi_l^0$ with $\phi_l = \xi_j(x,y,t) - \xi_i(x,y,t)$ in (3.5) for $l = \ell([i,j]) = 1,\ldots,4$,

$$\phi_1 = \xi_4 - \xi_1, \quad \phi_2 = \xi_3 - \xi_1, \quad \phi_3 = \xi_4 - \xi_2, \quad \phi_4 = \xi_3 - \xi_2,$$

One should note here that we have a linear relation among the phase functions ϕ_i's, i.e.

$$\phi_1 + \phi_4 = \phi_2 + \phi_3 = (\xi_3 + \xi_4) - (\xi_1 + \xi_2).$$

Then the last two terms in the τ-function (5.4) becomes

$$\left(C_{1,4}e^{\phi_1^0+\phi_4^0} + C_{2,3}e^{\phi_2^0+\phi_3^0}\right)e^{\phi_1+\phi_4} = (a_{1,3}a_{2,4} - a_{1,4}a_{2,3})\frac{\kappa_3-\kappa_4}{\kappa_1-\kappa_2}e^{\phi_1+\phi_4}.$$

This implies that for the regular soliton solution, we need to choose appropriate constants $\phi_1^0,\ldots,\phi_4^0$ so that $a_{1,3}a_{2,4} - a_{14}a_{2,3} \geq 0$, i.e. $A \in \mathrm{Gr}(2,4)_{\geq 0}$.

6. The Schottky uniformization

The main question in the present paper is to construct a smooth compact Riemann surface $\mathcal{R}_g$ associated with the KP soliton whose M-theta function $\widetilde{\vartheta}_g$ is obtained by taking a tropical (singular) limit of $\mathcal{R}_g$. We answer to this question using the Schottky uniformization theorem [6, 2]. A Schottky group is defined as a finitely generated, discontinuous subgroup of $SL_2(\mathbb{C})$ which are free and purely loxodromic [2]. In this paper, we consider a special case of the Schottky group, which is generated by purely hyperbolic Möbius transformations in $SL_2(\mathbb{R})$. It was shown in [6] that any compact Riemann surface $\mathcal{R}$ can be uniformized by the Schottky group Γ, which can be represented as

$$\mathcal{R} \;\cong\; \Omega(\Gamma)/\Gamma,$$

where $\Omega(\Gamma)$ is the set of discontinuity of Γ (see also [2]).

In order to define our Schottky group Γ_A for $A \in \mathrm{Gr}(N,M)_{\geq 0}$, we start with the following definition.

Definition 6.1. *For each element $[i,j] \in P_1(A)$, we define a pair of real numbers $\{\kappa_{i,j}, \kappa_{j,i}\}$ with the order,*

(a) $\kappa_k < \kappa_{k,\bullet} < \kappa_l < \kappa_{l,\bullet}$ *for all* $k < l \in [M]$, *and*

(b) $\kappa_{k,p} < \kappa_{k,q}$, *when* $p > q$ *and for* $k \in [M]$.

Let $\gamma_{[i,j]}$ be the hyperbolic Möbius transform on $\mathbb{CP}^1$ having two fixed points $\{\kappa_{i,j}, \kappa_{j,i}\}$, which is defined by

$$\frac{\gamma_{[i,j]}(z) - \kappa_{i,j}}{\gamma_{[i,j]}(z) - \kappa_{j,i}} = \mu_{i,j} \frac{z - \kappa_{i,j}}{z - \kappa_{j,i}}, \tag{6.1}$$

where $\mu_{i,j}$ is the multiplier which is symmetric real constant with $0 < \mu_{[i,j]} < 1$. Then the fixed points $\kappa_{i,j}$ and $\kappa_{j,i}$ are attractive and repulsive, respectively. Then we define the Schottky group Γ_A associated with $A \in \mathrm{Gr}(N, M)_{\geq 0}$ as a Fuchsian group given by

$$\Gamma_A := \langle\ \gamma_{[i,j]} \in PSL_2(\mathbb{R}) : [i,j] \in P_1(A)\ \rangle. \tag{6.2}$$

where $\gamma_{[i,j]}$ in (6.1) is expressed as

$$\gamma_{[i,j]} = \frac{1}{(\kappa_{i,j} - \kappa_{j,i})\sqrt{\mu_{i,j}}} \begin{pmatrix} \kappa_{i,j} - \mu_{i,j}\kappa_{j,i} & -\kappa_{i,j}\kappa_{j,i}(1 - \mu_{i,j}) \\ 1 - \mu_{i,j} & -(\kappa_{j,i} - \mu_{i,j}\kappa_{i,j}) \end{pmatrix}. \tag{6.3}$$

In Section 6.1 below, we directly construct $\gamma_{[i,j]}$ as a deformation of the singular curve (Riemann surface) associated with each element $A \in \mathrm{Gr}(N, M)_{\geq 0}$.

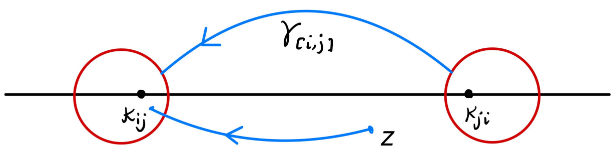
The isometric circle $I(\gamma_{[i,j]})$ of $\gamma_{[i,j]}$ in (6.3) is then given by

$$|(1 - \mu_{i,j})z - (\kappa_{i,j} - \mu_{i,j}\kappa_{j,i})| = (\kappa_{j,i} - \kappa_{i,j})\sqrt{\mu_{i,j}},$$

whose center and radius are

$$\text{Center} = \frac{\kappa_{i,j} - \mu_{i,j}\kappa_{j,i}}{1 - \mu_{i,j}}, \qquad \text{Radius} = \frac{\kappa_{j,i} - \kappa_{i,j}}{1 - \mu_{i,j}}\sqrt{\mu_{i,j}}. \tag{6.4}$$

Taking $\mu_{i,j}$ small enough, one can assume that all the isometric circles are disjoint. Note that $\gamma_{[i,j]}$ maps outside of the isometric circle $I(\gamma_{[i,j]}^{-1})$ into the interior of $I(\gamma_{[i,j]})$, see the figure below.



The (isometric) fundamental region, denoted by $\mathcal{F}(\Gamma_A)$, of Γ_A is given by $\mathbb{CP}^1$ with $2g$ holes of isometric circles, i.e.

$$\mathcal{F}(\Gamma_A) := \mathrm{Ext}\left(\bigcup_{[i,j]\in P_1(A)} \overline{\mathrm{Int}\ \left(I(\gamma_{[i,j]})\right) \cup \mathrm{Int}\ \left(I(\gamma_{[i,j]}^{-1})\right)} \right), \tag{6.5}$$

where $\mathrm{Ext}(D)$ means the set of exterior points of the set D, and $\mathrm{Int}(I(\gamma))$ represents the interior points of the isometric circle $I(\gamma)$.

For each $[i,j] \in P_1(A)$, let $\omega_{[i,j]}$ be the differentials on $\Omega(\Gamma_A)$, the set of discontinuity of Γ_A, defined by

$$\omega_{[i,j]} = \frac{dz}{2\pi i} \sum_{\gamma\in\Gamma_A/\langle\gamma_{[i,j]}\rangle} \left(\frac{1}{z - \gamma(\kappa_{i,j})} - \frac{1}{z - \gamma(\kappa_{j,i})} \right), \tag{6.6}$$

where γ runs through all representatives of the right coset classes of Γ_A by its cyclic subgroup $\langle \gamma_{[i,j]} \rangle$ generated by $\gamma_{[i,j]}$. Here $\Omega(\Gamma_A)$ can be expressed as $\Omega(\Gamma_A) = \cup_{\gamma \in \Gamma_A} \gamma(\mathcal{F}(\Gamma_A))$. It is also known [6, 2] that the infinite sum in (6.6) converges absolutely for sufficiently small $\mu_{i,j}$. Then we have the lemma.

Lemma 6.1. *The differentials $\omega_{[i,j]}$ are holomorphic on $\Omega(\Gamma_A)$,*

$$\omega_{[i,j]}(z) = \omega_{[i,j]}(\gamma(z)) \qquad \textit{for any} \quad \gamma \in \Gamma_A.$$

Proof. Let α be a differential given by

$$\alpha(z) = \left(\frac{1}{z-A} - \frac{1}{z-B} \right) dz = \frac{A-B}{(z-A)(z-B)}\, dz.$$

Then for $\sigma \in \Gamma_A$, we have

$$\alpha(\sigma(z)) = \frac{\sigma^{-1}(A) - \sigma^{-1}(B)}{(z-\sigma^{-1}(A))(z-\sigma^{-1}(B))}\, dz.$$

Then taking $A = \gamma(\kappa_{i,j})$ and $B = \gamma(\kappa_{j,i})$, and then $\sigma^{-1}\gamma \in \Gamma_A/\langle \gamma_{[i,j]} \rangle$. Summing over all the element in $\Gamma_A/\langle \gamma_{[i,j]} \rangle$ gives a proof. □

Then we have the following proposition.

Proposition 6.1. *The period integrals of the differentials are given by*

$$\oint_{a_{[i,j]}} \omega_{[k,l]} = \begin{cases} 1, & \textit{if} \ \ [i,j] = [k,l], \\ 0, & \textit{if} \ \ [i,j] \neq [k,l]. \end{cases}$$

$$\oint_{b_{[i,j]}} \omega_{[k,l]} = \frac{1}{2\pi i} \sum_{\gamma \in \langle \gamma_{[i,j]} \rangle \backslash \Gamma_A / \langle \gamma_{[k,l]} \rangle} \ln\, [\kappa_{i,j}, \kappa_{j,i}; \gamma(\kappa_{k,l}), \gamma(\kappa_{l,k})], \qquad (6.7)$$

where $[\kappa_{i,j}, \kappa_{j,i}; \gamma(\kappa_{k,l}), \gamma(\kappa_{l,k})]$ is the cross ratio given by

$$[\kappa_{i,j}, \kappa_{j,i}; \gamma(\kappa_{k,l}), \gamma(\kappa_{l,k})] := \frac{(\kappa_{i,j} - \gamma(\kappa_{k,l}))(\kappa_{j,i} - \gamma(\kappa_{l,k}))}{(\kappa_{i,j} - \gamma(\kappa_{l,k}))(\kappa_{j,i} - \gamma(\kappa_{k,l}))},$$

which takes $\mu_{i,j}$ when $[i,j] = [k,l]$ and $\gamma \in \langle \gamma_{[i,j]} \rangle$.

Proof. The period integral over $a_{[i,j]}$ are obvious, and this implies that $\omega_{[i,j]}$ is normalized. The integral over $b_{[i,j]}$ gives a period integral over b-cycle. For a point a on the isometric circle $I(\gamma_{[i,j]}^{-1})$, i.e. $\gamma_{[i,j]}(a) \in I(\gamma_{[i,j]})$, the integral gives

$$\begin{aligned} \int_{b_{[i,j]}} \omega_{[k,l]} &= \frac{1}{2\pi i} \sum_{\gamma \in \Gamma_A/\langle \gamma_{[k,l]} \rangle} \ln \left. \frac{z - \gamma(\kappa_{k,l})}{z - \gamma(\kappa_{l,k})} \right|_a^{\gamma_{[i,j]}(a)} \\ &= \frac{1}{2\pi i} \sum_{\gamma \in \Gamma_A/\langle \gamma_{[k,l]} \rangle} \ln \frac{(\gamma_{[i,j]}(a) - \gamma(\kappa_{k,l}))(a - \gamma(\kappa_{l,k}))}{(\gamma_{[i,j]}(a) - \gamma(\kappa_{l,k}))(a - \gamma(\kappa_{k,l}))}. \end{aligned}$$

Here, if $[i,j]=[k,l]$ and $\gamma\in\langle\gamma_{[i,j]}\rangle$, then by (6.1),

$$\frac{(\gamma_{[i,j]}(a)-\gamma(\kappa_{k,l}))(a-\gamma(\kappa_{l,k}))}{(\gamma_{[i,j]}(a)-\gamma(\kappa_{l,k}))(a-\gamma(\kappa_{k,l}))}=\frac{(\gamma_{[i,j]}(a)-\kappa_{i,j})(a-\kappa_{j,i})}{(\gamma_{[i,j]}(a)-\kappa_{j,i})(a-\kappa_{i,j})}=\mu_{i,j}.$$

Since $\lim_{n\to\infty}\gamma_{[i,j]}^{n}(a)=\kappa_{i,j}$, $\lim_{n\to\infty}\gamma_{[i,j]}^{-n}(a)=\kappa_{j,i}$, if $[i,j]\neq[k,l]$ or $\gamma\notin\langle\gamma_{[i,j]}\rangle$, then

$$\begin{aligned}&\prod_{n\in\mathbb{Z}}\left(\frac{(\gamma_{[i,j]}(a)-\gamma_{[i,j]}^{-n}\gamma(\kappa_{k,l}))(a-\gamma_{[i,j]}^{-n}\gamma(\kappa_{l,k}))}{(\gamma_{[i,j]}(a)-\gamma_{[i,j]}^{-n}\gamma(\kappa_{l,k}))(a-\gamma_{[i,j]}^{-n}\gamma(\kappa_{k,l}))}\right)\\ &=\prod_{n\in\mathbb{Z}}\left(\frac{(\gamma_{[i,j]}^{n+1}(a)-\gamma(\kappa_{k,l}))(\gamma_{[i,j]}^{n}(a)-\gamma(\kappa_{l,k}))}{(\gamma_{[i,j]}^{n+1}(a)-\gamma(\kappa_{l,k}))(\gamma_{[i,j]}^{n}(a)-\gamma(\kappa_{k,l}))}\right)\\ &=\frac{(\kappa_{i,j}-\gamma(\kappa_{k,l}))(\kappa_{j,i}-\gamma(\kappa_{l,k}))}{(\kappa_{i,j}-\gamma(\kappa_{l,k}))(\kappa_{j,i}-\gamma(\kappa_{k,l}))}\end{aligned}$$

which completes the proof. □

As the summary of these results, we now give the main theorem.

Theorem 6.2. *Given irreducible $A\in Gr(N,M)_{\geq 0}$, a real compact Riemann surface $\mathcal{R}_g$ can be constructed by the Schottky group Γ_A defined in* (6.2) *with* (6.3), *i.e.*

$$\mathcal{R}_g\cong\Omega(\Gamma_A)/\Gamma_A,$$

where $g=|P_1(A)|$ in (4.3) *and $\Omega(\Gamma_A)$ is the set of discontinuity of Γ_A. The ϑ-function defined on $\mathcal{R}_g$ is given by* (2.2) *with the period matrix in* (6.7).

6.1. From TNN Grassmannians to graphs

In this section, we explain how one can construct the Schottky group by deforming a singular curve associated with an element $A\in\mathrm{Gr}(N,M)_{\geq 0}$ for the KP soliton.

Let us first define an oriented graph $\Delta_A(V,E)$ associated with the element $A\in\mathrm{Gr}(N,M)_{\geq 0}$, whose the set of vertices V and the set of oriented edges E are given as follows:

(a) $V:=\{v_0,v_k\,(k\in[M])\}$,

(b) $E:=\{e_k\,(k\in[M]),\ e_{[i,j]}\,([i,j]\in P_1(A)\}$,

where each edge e_k is from v_0 to v_k, and $e_{[i,j]}$ from v_i to v_j. Then the set of closed paths $e_i\cdot e_{[i,j]}\cdot e_j^{-1}$ forms the fundamental group $\pi_1(\Delta_A,v_0)$ with the base point v_0. The homological group $H_1(\Delta_A;\mathbb{Z})$ is then given by abelianization of π and the dimension is $\dim H_1(\Delta,\mathbb{Z})=|P_1(A)|$. Note that these closed paths are related to the $b_{[i,j]}$-cycles defined in the Ⅎ-diagram (see Section 4).

We call algebraic curves defined over $\mathbb{R}$ *real curves*, and construct a singular real curve $\mathcal{C}_A$ with dual graph Δ_A and a family of real curves $\mathcal{R}_A$ as deformations of $\mathcal{C}_A$. Denote by $\mathbb{RP}^1$ the real projective line $\mathbb{R}\cup\{\infty\}$ which is identified with an oriented circle according to the increase of real numbers. Put $\mathcal{P}_{v_0}=\mathbb{RP}^1$ with counter-clockwise orientation, and take points κ_k $(k\in[M])$ on $\mathcal{P}_{v_0}\setminus\{\infty\}$ with the ordering (3.6).

For each vertex v_k ($k \in [M]$), put $\mathcal{P}_{v_k} = \mathbb{RP}^1$ with counter-clockwise orientation, and take points $\lambda_k \in \mathcal{P}_{v_k} \setminus \{\infty\}$ and $\lambda_{k,l} \in \mathcal{P}_{v_k} \setminus \{\infty\}$ if $[k,l] \in P_1(A)$ or $[l,k] \in P_1(A)$ such that $\lambda_{k,l} < \lambda_k$ and $\lambda_{k,l} < \lambda_{k,m}$ for $l > m$. Then the singular real curve $\mathcal{C}_A$ with dual graph Δ_A is obtained as a union of $\mathcal{P}_{v_0}$ and $\mathcal{P}_{v_k}$ ($k \in [M]$) by identifying

$$\kappa_k = \lambda_k \ (k \in [M]), \qquad \lambda_{i,j} = \lambda_{j,i} \ ([i,j] \in P_1(A)),$$

and hence the (arithmetic) genus of $\mathcal{C}_A$ is $g = |P_1(A)|$. For small positive parameters ν_k ($k \in [M]$) and $\nu_{i,j} = \nu_{j,i}$ ($[i,j] \in P_1(A)$), let $\mathcal{R}_A$ be a family of real curves as deformations of $\mathcal{C}_A$ obtained by gluing

$$\mathcal{C}_A \setminus \{\text{neighborhoods of singular points}\}$$

under the relations

$$(z_0 - \kappa_k)(z_k - \lambda_k) = -\nu_k, \tag{6.5}$$

and

$$(z_i - \lambda_{i,j})(z_j - \lambda_{j,i}) = -\nu_{i,j}, \tag{6.6}$$

where z_i are the coordinates of $\mathcal{P}_{v_i}$. By these relations, for $[i,j] \in P_1(A)$, if $z, w \in \mathcal{P}_{v_0} = \mathbb{RP}^1$ are related as

$$z \in \mathcal{P}_{v_0} \quad \overset{(6.5)}{\longmapsto} \quad z_i \in \mathcal{P}_{v_i} \quad \overset{(6.6)}{\longmapsto} \quad z_j \in \mathcal{P}_{v_j} \quad \overset{(6.5)}{\longmapsto} \quad w \in \mathcal{P}_{v_0},$$

then we have

$$w - \kappa_j = -\frac{\nu_j}{z_j - \lambda_j} = \frac{a\nu_j(z - \kappa_i) - \nu_i\nu_j}{(ab + \nu_{i,j})(z - \kappa_i) - bs_i}$$

where $a = \lambda_i - \lambda_{i,j}$ and $b = \lambda_j - \lambda_{j,i}$. This gives the Möbius transform $\gamma : z \mapsto w = \gamma(z)$ on $\mathcal{P}_{v_0}$ with $\gamma \in PSL_2(\mathbb{R})$,

$$\gamma = \frac{1}{\sqrt{\nu_i\nu_j\nu_{i,j}}} \begin{pmatrix} c\kappa_j + a\nu_j & -c\kappa_i\kappa_j - \nu_i\nu_j - a\kappa_i\nu_j - b\kappa_j\nu_i \\ c & -c\kappa_i - b\nu_i \end{pmatrix},$$

where $c = ab + \nu_{i,j}$. Then introducing the Schottky parameters $\{\kappa_{i,j}, \kappa_{j,i}, \mu_{i,j}\}$ in terms of $\{a\nu_j, b\nu_i, c\}$, we have $\gamma = \gamma_{[i,j]}$ defined in (6.3). We can also see

$$\kappa_{k,l} - \kappa_k = \Theta(\nu_k), \qquad \mu_{i,j} = \Theta(\nu_i\nu_{i,j}\nu_j),$$

where $f = \Theta(g)$ means that there exists positive constants c_1, c_2 satisfying $c_1|g| \leq |f| \leq c_2|g|$ asymptotically. Therefore, $\mathcal{R}_A$ with sufficiently small $\nu_k, \nu_{i,j} > 0$ gives a family of real curves which are Schottky uniformized by real Schottky groups Γ_A with free generators $\gamma_{[i,j]}$ ($[i,j] \in P_1(A)$). Furthermore, under $\nu_k, \nu_{i,j} \to 0$, $\kappa_{i,j} \to \kappa_i$, $\kappa_{j,i} \to \kappa_j$ and $\gamma(\kappa_{i,j}) - \gamma(\kappa_{j,i}) \to 0$ for any $\gamma \in (\Gamma_A \setminus \langle\gamma_{[i,j]}\rangle)/\langle\gamma_{[i,j]}\rangle$. Thereore, the differentials $\omega_{[i,j]}$ given in (6.4) has the limit

$$\omega_{[i,j]} \quad \longrightarrow \quad \frac{dz}{2\pi i}\left(\frac{1}{z - \kappa_i} - \frac{1}{z - \kappa_j}\right),$$

and by Proposition 6.1, the period matrix has the limit

$$\exp\left(2\pi i \oint_{b_{[i,j]}} \omega_{[k,l]}\right) \longrightarrow \begin{cases} 0 & (i=k \text{ or } j=l), \\ [\kappa_i, \kappa_j; \kappa_k, \kappa_l] & (i \neq k \text{ and } j \neq l). \end{cases}$$

Taking appropriate pairs $\{\alpha_j, \beta_j\}$ in the normalization in Section 2.1, we recover the limits in (2.4) and (2.5).

6.2. Quasi-periodic solutions

In this section, we just recall [2] that a quasi-periodic solution can be obtained by the theta function (2.2) using the Schottky group. In [2] (Section 5.5 in p.160), the solution $u(x, y, t)$ of the KP equation is given by

$$u(x,y,t) = 2\,\partial_x^2 \ln \vartheta_g(\mathbf{U}^1 x + \mathbf{U}^2 y + \mathbf{U}^3 t + \mathbf{D};\, \Omega_A) + 2C$$

where $\mathbf{U}^k = (U^k_{[i,j]} : [i,j] \in P_1(A))$ for $k = 1, 2, 3$ are g-dimensional vectors given by

$$U^k_{[i,j]} := \sum_{\gamma \in \Gamma_A / \langle \gamma_{[i,j]} \rangle} \left(\gamma(\kappa_{i,j})^k - \gamma(\kappa_{j,i})^k \right).$$

The period matrix Ω_A is given by (6.7), and $\mathbf{D}$ is an arbitrary constant vector. The constant C is computed as

$$C = \sum_{[i,j] \in P_1(A)} \left(\frac{(\kappa_{j,i} - \kappa_{i,j})\sqrt{\mu_{i,j}}}{1 - \mu_{i.j}} \right)^2.$$

Now it is easy to confirm that the solution $u(x, y, t)$ leads to the KP soliton in the limit with $\kappa_{i,j} \to \kappa_i$, $\kappa_{j,i} \to \kappa_j$ and $\mu_{i,j} \to 0$.

Remark 6.3. *In general, our construction of a real compact Riemann surface $\mathcal{R}$ does not give the so-called* M*-curve* [4]*, which requires that on $\mathcal{R}$, the involution σ must have a maximum number of orvals chosen from the homological basis. Here the involution σ acts on $H_1(\mathcal{R}; \mathbb{Z}) = \langle a_j, b_j; j = 1, \ldots, g \rangle$ by*

$$\sigma(a_j) = a_j, \quad \sigma(b_j) = -b_j, \qquad \text{for } j = 1, \ldots, g.$$

In the case that the Riemann surface is not an M*-curve, the quasi-periodic solution of the KP equation is not regular* [4] *(Theorem in p.271). We will discuss in more details in a forth-coming paper* [11].

7. Examples

Here we give two examples, and show the fundamental domains $\mathcal{F}(\Gamma_A)$.

7.1. The cases of $\mathbf{Gr}(2,4)_{\geq 0}$

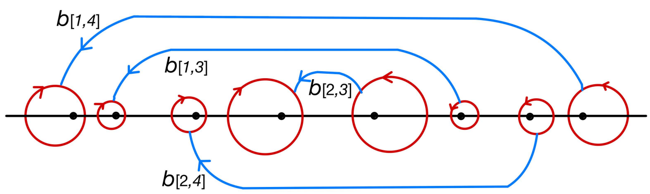
(a) **The cases with $g = 4$:** Consider the case with the $O\text{⅃}$-diagram $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$. Then we have

$$P_1(A) = \{[1,4], [1,3], [2,4], [2,3]\}, \quad \text{i.e.} \quad g = 4.$$

The element $\gamma_{[i,k]}$ in the Schottky group Γ_A are defined by (6.3), where

$$\kappa_{1,4} \;<\; \kappa_{1,3} \;<\; \kappa_{2,4} \;<\; \kappa_{2,3} \;<\; \kappa_{3,2} \;<\; \kappa_{3,1} \;<\; \kappa_{4,2} \;<\; \kappa_{4,1}.$$

The fundamental domain $\mathcal{F}(\Gamma_A)$ is shown in the figure below, that is, $\mathcal{F}(\Gamma_A)$ is the domain outside the isometric circles. In the figure, the dots on the real line are $\kappa_{k,l}$, and the b-cycles show the actions of the group elements $\gamma_{i,j]}$ for $[i,j] \in P_1(A)$.



We consider the limit $\mu_{i,j} \to 0$ but keep all $\kappa_{k.l}$ distinct. Then the limit gives a 4-soliton solution of Hirota-type (see e.g. [7]), i.e. 4 line solitons without resonance. However, this solution is *not* regular as one can see from the matrix $\tilde{A}$ obtained by the limit, i.e. $\widetilde{A} \notin \mathrm{Gr}(4,8)_{\geq 0}$ [8],

$$\widetilde{A} = \begin{pmatrix} 1 & & & & & & & a_{[1,4]} \\ & 1 & & & & a_{[1,3]} & & \\ & & 1 & & & & a_{[2,4]} & \\ & & & 1 & a_{[2,3]} & & & \end{pmatrix}$$

where $a_{[i,j]}$ are nonzero constants, and all other entries except pivots are zero. The corresponding M-theta function can be computed by follwoing Section 5. Then taking further limits $\kappa_{i,j} \to \kappa_i$ and $\kappa_{j,i} \to \kappa_j$, we obtain the regular solution with

$$A = \begin{pmatrix} 1 & 0 & a_{1,3} & a_{1,4} \\ 0 & 1 & a_{2,3} & a_{2,4} \end{pmatrix}$$

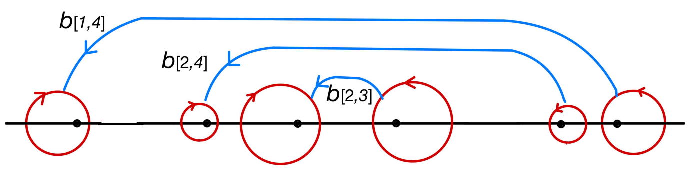
where $a_{1,3}, a_{1,4} < 0$, $a_{2,3}, a_{2,4} > 0$ and $a_{13}a_{2,4} - a_{2,3}a_{1,4} \geq 0$ for $A \in \mathrm{Gr}(2,4)_{\geq 0}$

(b) **A case with $g = 3$:** Consider the OI-diagram $\begin{array}{|c|c|}\hline 1 & \\ \hline 2 & 3 \\ \hline\end{array}$, which gives

$$P_1(A) = \{[1,4], [2,4], [2,3]\}.$$

The Schottky parameters $\{\kappa_{i,j}; [i,j] \in P_1(A)\}$ are given by

$$\kappa_{1,4} \;<\; \kappa_{2,4} \;<\; \kappa_{2,3} \;<\; \kappa_{3,1} \;<\; \kappa_{4,2} \;<\; \kappa_{4,1}.$$



The limit with $\mu_{i,j} \to 0$ (keeping $\kappa_{i,j}$ distinct) gives the matrix

$$\widetilde{A} = \begin{pmatrix} 1 & & & & & a_{[1,4]} \\ & 1 & & & a_{[2,4]} & \\ & & 1 & a_{[2,3]} & & \end{pmatrix}$$

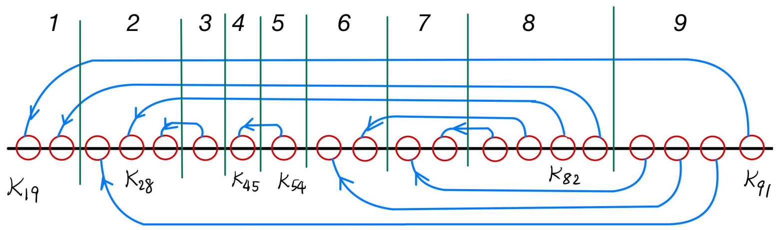
which gives a 3-soliton solution without resonance (i.e. Hirota-type), and it is regular if $a_{[1,4]} > 0$, $a_{[2,4]} < 0$ and $a_{[2,3]} > 0$. The corresponding matrix $A \in \mathrm{Gr}(2,4)_{\geq 0}$ is

$$A = \begin{pmatrix} 1 & 0 & 0 & a_{1,4} \\ 0 & 1 & a_{2,3} & a_{2,4} \end{pmatrix}$$

where $a_{1,4} < 0$ and $a_{2,3}, a_{2,4} > 0$. We also note that the quasi-periodic solution is regular, and the Riemann surface in this case is an M-curve of genus 3.

7.2. A case in $\mathbf{Gr}(5,9)_{\geq 0}$

Here we just illustrate the fundamental domain $\mathcal{F}(\Gamma_A)$ for Example 4.4 (see the figure below). The quasi-periodic solution associated with the Riemann surface uniformized by the Schottky group may not be regular.



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