Deformation of a generically finite map to a hypersurface embedding and the moduli space of smooth hypersurfaces in abelian varieties

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Abstract

In this presentation, we give a structure theorem for projective manifolds W_0 with the property of admitting a 1-parameter deformation where W_t is a hypersurface in a projective smooth manifold Z_t. Their structure is the one of special iterated univariate coverings which we call of normal type. We give an application to the case where Z_t is a projective space, respectively an abelian variety. We also give a characterizaton of smooth ample hypersurfaces in abelian varieties and describe an irreducible connected component of their moduli space. All works in this presentation have been carried out by the joint research with Fabrizio Catanese. This presenation is based on two papers [5], [6].

Many years ago Sernesi [9] showed that small deformations of complete intersections in projective space, of dimension $n \geq 2$ (the case of curves, $n = 1$ is of quite different nature), are again complete intersections, unless the complete intersection defines a K3 surface (i.e., $n = 2$ and the canonical bundle is trivial). Hence, in particular, smooth hypersurfaces in projective space $\mathbb{P}^{n+1}$ form an open set in the Kuranishi space, respectively an open set in the moduli space when they are of general type, unless $n = 2$ and the degree equals 4. In considering the closure of this set in the moduli space, we have to deal with varieties W_0 of the same dimension, given together with a generically finite rational map $\phi_0 : W_0 \dashrightarrow \mathbb{P}^{n+1}$.

As shown by Horikawa in [7], already in the easiest nontrivial case $n = 2$, $\deg(W_0) = 5$ the situation becomes rather complicated. But we show here that things are simpler in the case where ϕ_0 is a morphism.

A similar result to Sernesi's holds for hypersurfaces in an Abelian variety (Kodaira and Spencer's theorem 14.4 in [8]), and we can consider the closure of the locus of hypersurfaces X in Abelian varieties (for $n \geq 2$ the Abelian variety is just the Albanese variety of X) observing that in this case any limit W_0 has a generically finite Albanese map $\phi_0 : W_0 \to A_0$ (see for instance Lemma 149 of [4]). Also in this case we can ask the question of characterizing the morphisms ϕ_0 admitting a deformation which is a hypersurface embedding in some Abelian variety, deformation of the original one.

To illustrate our main result, let us consider two simple examples, the first one where the image of W_0 is the smooth hypersurface $X := \{\sigma = 0\} \subset \mathbb{P}^{n+1}$, σ being a homogeneous polynomial of degree d. We let then W_0 be the complete intersection in the weighted projective space $\mathbb{P}(1, 1, \ldots, 1, d)$ defined by the equations

$$W_0 = \{(z_0, z_1, \ldots, z_{n+1}, w) | \sigma(z_0, z_1, \ldots, z_{n+1}) = 0,$$
$$P(z_0, z_1, \ldots, z_{n+1}, w) := w^m + \sum_{i=1}^{m} w^{m-i} a_i(z_0, z_1, \ldots, z_{n+1}) = 0\}. \quad (1)$$

We can easily deform the complete intersection by deforming the degree d equation adding a constant times the variable w, hence obtaining the following complete intersection:

$$P(z_0, z_1, \ldots, z_{n+1}, w) = 0, tw - \sigma(z_0, z_1, \ldots, z_{n+1}) = 0, t \in \mathbb{C}.$$

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Clearly, for $t = 0$ we obtain the previous W_0, a degree m covering of the hypersurface $X = \{\sigma = 0\}$, whereas for $t \neq 0$ we can eliminate the variable w and obtain a hypersurface W_t in $\mathbb{P}^{n+1}$ with equation (of degree md)

$$P(z_0, z_1, \dots, z_{n+1}, \sigma(z)/t) = 0.$$

Example 0.1 (Iterated weighted deformations).
Now, one can iterate this process, and consider, in the weighted projective space

$$\mathbb{P}(1, 1, \dots, 1, d, dm_1, \dots, dm_k), \quad m_1 | m_2 | \dots | m_k,$$

a complete intersection W of multidegrees $(d, dm_1, \dots, dm_k, dm)$,
where $m_k | m =: m_{k+1}$.

Then, necessarily, there exist constants $t_0, t_1, \dots, t_k$ such that the equations of W have the following form, where the Q_j's are general weighted homogeneous polynomials of degree $= dm_j$ (in particular we assume them to be monic, so that the rational map to projective space is a morphism):

$$\begin{cases} \sigma(z) = w_0 t_0 \\ Q_1(w_0, z) = w_1 t_1 \\ \dots \quad \dots \\ Q_k(w_0, \dots, w_{k-1}, z) = w_k t_k \\ Q_{k+1}(w_0, \dots, w_k, z) = 0. \end{cases} \tag{2}$$

Again, if all the t_j's are $\neq 0$, we can eliminate the variables w_j, and we obtain a hypersurface in $\mathbb{P}^{n+1}$.

To generalize the above description, we need to introduce the following terminology.

Definition 0.2
i) Given a complex space (or a scheme) X, a **univariate covering** *of X is a hypersurface Y, contained in a line bundle over X, and defined there as the zero set of a monic polynomial.*

This means, $Y = \underline{Spec}(\mathcal{R})$, where $\mathcal{R}$ is the quotient algebra of the symmetric algebra over an invertible sheaf $\mathcal{L}$, $Sym(\mathcal{L}) = \oplus_{i \geq 0} \mathcal{L}^{\otimes i}$, by a monic (univariate) polynomial:

$$\mathcal{R} := Sym(\mathcal{L})/(P), P = w^m + a_1(x) w^{m-1} + a_2(x) w^{m-2} + \dots + a_m(x).$$

Here $a_j \in H^0(X, \mathcal{L}^{\otimes j})$. The univariate covering is said to be **smooth** *if both X and Y are smooth.*

ii) An **iterated univariate covering** *$W \to X$ is a composition of univariate coverings*

$$f_{k+1} : W \to X_k, f_k : X_k \to X_{k-1}, \dots, f_1 : X_1 \to X,$$

whose associated line bundles are denoted $\mathcal{L}_k, \mathcal{L}_{k-1}, \dots, \mathcal{L}_1, \mathcal{L}_0$.

iii) In the case where $X \subset Z$ is a (smooth) hypersurface, we say that the iterated univariate covering is of **normal type** *if*

- *all the line bundles $\mathcal{L}_j$ are pull back from X of a line bundle of the form $\mathcal{O}_X(m_j X)$, and moreover*
- *$m_1 | m_2 | \dots | m_k$, and the degree of f_j equals $\frac{m_j}{m_{j-1}}$.*

- *we say that the iterated covering is* **normally induced** *if moreover all the coefficients* $a_I(x)$ *of the polynomials*

$$Q_j(w_0, \dots, w_{j-1}, x) = \sum_I a_I(x) w^I$$

describing the intermediate extensions are sections of a line bundle $\mathcal{O}_X(r(I)X)$ *coming from* $H^0(Z, \mathcal{O}_Z(r(I)X))$.

Remark 0.3 *The property that the iterated univariate covering* $W \to X$ *is normally induced clearly means that it is the restriction to* X *of an iterated univariate covering of* Z.

The property that the former is smooth does not necessarily imply that also the latter is smooth.

Definition 0.4 A 1-parameter deformation to hypersurface embedding *consists of the following data:*

1. *a one dimensional family of smooth projective varieties of dimension* n *(i.e., a smooth projective holomorphic map* $p : \mathcal{W} \to T$ *where* T *is a germ of a smooth holomorphic curve at a point* $0 \in T$*) mapping to another family* $\pi : \mathcal{Z} \to T$ *of smooth projective varieties of dimension* $n+1$ *via a relative map* $\Phi : \mathcal{W} \to \mathcal{Z}$ *such that* $\pi \circ \Phi = p$ *(hence we have the following commutative diagram)*

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\Phi} & \mathcal{Z} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & T, & \end{array}$$

such that moreover

2. *for* $t \neq 0$ *in* T, Φ_t *is an embedding of* $W_t := p^{-1}(t)$ *in* Z_t,

3. *the restriction of the map* Φ *on* W_0 *is a generically finite morphism of degree* m*, so that the image of* $\Phi|_{W_0}$ *is the cycle* $\Sigma_0 := mX$ *where* X *is a reduced hypersurface in* Z_0*, defined by an equation* $X = \{\sigma = 0\}$.

Put in concrete terms, one can take a local coordinate t for T at 0, and write, locally around $\{t = 0\}$ the equation of the image $\Sigma := \Phi(\mathcal{W})$ in $\mathcal{Z}$ via the Taylor series development in t, in terms of local coordinates $z = (z_1, \dots, z_n, z_{n+1})$ on Z_0,

$$\Sigma(z,t) : \sigma(z)^m + t\sigma_1(z) + t^2\sigma_2(z) + \dots + t^{m-1}\sigma_{m-1}(z) + \dots = 0 \ (\star).$$

$\mathcal{W}$ is a resolution of Σ and the next theorem indicates exactly the sequence of blow-ups needed in order to obtain the resolution $\mathcal{W}$.

Theorem 0.5 *(A) Suppose we have a 1-parameter deformation to hypersurface embedding*

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\Phi} & \mathcal{Z} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & T. & \end{array}$$

and assume that K_{W_0} *is ample.*
Then we have:
(A1) X *is smooth,*

(A2) There are line bundles $\mathcal{L}_0, \dots, \mathcal{L}_k$ on $\mathcal{Z}$, such that $\mathcal{L}_j|_{Z_0} = \mathcal{O}_{Z_0}(m_j X)$ for $j = 0, \dots, k$, with $1 = m_0 | m_1 | m_2 \dots | m_k | m_{k+1} := m$ (here m is the degree of the morphism $\Phi_0 : W_0 \to X$), and such that W_0 is a complete intersection in $\mathcal{L}_0 \oplus \dots \oplus \mathcal{L}_k|_{Z_0}$, with Φ_0 a normally induced iterated smooth univariate covering .

(A3) $\mathcal{W}$ is obtained from $\Sigma := \Phi(\mathcal{W})$ by a finite sequence of blow-ups.

Moreover the local equations of $\mathcal{W}$ are of the following standard form

$$\begin{cases} \sigma(z) = w_0 t \\ Q_1(w_0, z) = w_1 t \\ \cdots \quad \cdots \\ Q_k(w_0, \dots, w_{k-1}, z) = w_k t \\ Q_{k+1}(w_0, \dots, w_k, z, t) = 0. \end{cases} \tag{3}$$

(B1) Conversely, take any smooth iterated univariate covering of normal type

$$\phi_0 : W_0 \to X \subset Z_0$$

and take any 1-parameter family $\mathcal{Z}$ of deformations of Z_0.

Then the line bundle $\mathcal{O}_{Z_0}(X)$ extends to a line bundle $\mathcal{L}_0$ on the whole family $\mathcal{Z}$. And W_0 deforms to a hypersurface embedding if, for all $i \geq 2$, every section in $H^0(Z_0, \mathcal{O}_{Z_0}(iX))$ and every section in $H^0(X, \mathcal{O}_X(iX))$ comes from a section in $H^0(\mathcal{Z}, \mathcal{L}_0^{\otimes i})$.

(B2) This holds in particular, when the family $\mathcal{Z}$ is trivial, $\mathcal{Z} \cong Z_0 \times T$, if the necessary condition of being normally induced is fulfilled.

Remark 0.6 *(b1) More precisely, in (B1) above, there is a family $\mathcal{W}$ such that $\mathcal{W}$ is a complete intersection in $\mathcal{L}_0 \oplus \dots \oplus \mathcal{L}_k$ ($\mathcal{L}_i = \mathcal{L}_0^{\otimes m_i}$), and $\mathcal{W}$ is given as above; moreover, for $t \neq 0$ in T, the morphism Φ_t, induced on W_t by the bundle projection on Z_t, is an embedding.*

(b2) sufficient condition in (B2) is the surjectivity of $H^0(Z_0, \mathcal{O}_{Z_0}(iX)) \to H^0(X, \mathcal{O}_X(iX))$ for $i \geq 2$; this is implied by $H^1(Z_0, \mathcal{O}_{Z_0}(iX)) = 0, \forall i \geq 1$.

Remark 0.7 *The line bundle $\mathcal{O}_{Z_0}(X)$ extends to a line bundle $\mathcal{L}_0$ on the whole family $\mathcal{Z}$, because of the Lefschetz (1,1) theorem, since $\mathcal{O}_{Z_0}(mX)$ does.*

Observe moreover that there is a (non-canonical) isomorphism

$$\mathrm{Pic}^0(Z_0) \cong \mathrm{Pic}^0(Z_t),$$

whereas in general there is no isomorphism of $\mathrm{Pic}(Z_0)$ with $\mathrm{Pic}(Z_t)$.

The following two lemmas, and the fact that the resolution of a plane curve is obtained by a finite sequence of blow-ups, play an important role in the proof of theorem 0.5.

Lemma 0.8 *Suppose we have a one dimensional smooth family $p : \mathcal{W} \to T$ of smooth projective varieties of dimension n mapping to another flat family $q : \mathcal{Y} \to T$ of projective varieties of the same dimension via a relative map $\Psi : \mathcal{W} \to \mathcal{Y}$ over a smooth holomorphic curve T such that $q \circ \Psi = p$, i.e. we have the commutative diagram*

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\ \Psi\ } & \mathcal{Y} \\ & {}_{p}\searrow \quad \swarrow_{q} & \\ & T. & \end{array}$$

Assume that

1. *$\mathcal{Y}$ is normal and Gorenstein,*

2. Ψ is birational,

3. for $t \neq 0$ in T, Ψ induces an isomorphism,

4. K_{W_0} is ample.

Then we have that Ψ is an isomorphism, in particular $W_0 \cong Y_0$.

Lemma 0.9 *In the hypotheses of theorem 0.5, we have that $\sigma^{m-i} \,|\, \sigma_i$ for $i = 1, \dots, m-1$ in the equation $(\star)$.*

We make use the above theorem to characterize the deformations of morphisms to hypersurface embeddings. This result is particularly suitable in order to analyze when does the Albanese map deform to a hypersurface embedding.

Thus we can study the moduli space of compact Kähler manifolds diffeomorphic to ample hypersurfaces in Abelian varieties, essentially showing that we get a connected component of the moduli space once we add to the Hypersurfaces of a given dimension n, and of a given polarization type, the iterated univariate coverings of normal type.

More precisely, we have the following main results that characterize smooth ample hypersurfaces in Abelian varieties.

Let A be an Abelian variety of dimension $n+1$, and let $X \subset A$ be a smooth and ample divisor, whose Chern class is a polarization of type $(d_1, d_2, \dots, d_n, d_{n+1})$, where $d_i | d_{i+1}$, $\forall i = 1, \dots, n$.

We assume throughout that $n = dim(X) \geq 2$, so that Lefschetz' theorem says that

1. $\pi_1(X) \cong \pi_1(A) \cong \mathbb{Z}^{2n+2} =: \Gamma$;
2. $\Lambda^i(\Gamma) = H^i(A, \mathbb{Z}) \to H^i(X, \mathbb{Z})$ is an isomorphism for $i \leq n-1$, and is injective for $i = n$;
3. $H_i(X, \mathbb{Z}) \to H_i(A, \mathbb{Z})$ is an isomorphism for $i \leq n-1$, and is surjective for $i = n$.

We consider now a projective manifold which is diffeomorphic to X, actually some weaker hypotheses are sufficient:

- (a) Assume that Y is a complex projective manifold, or
- (a') Assume that Y is a **cKM = compact Kähler Manifold**, and that
- (b) Y is homotopically equivalent to X, or
- (b1) there is an isomorphism $\alpha_Y : \pi_1(Y) \to \Gamma$ and an isomorphism $\psi : H^*(Y, \mathbb{Z}) \cong H^*(X, \mathbb{Z})$ such that, letting $\alpha_X : \pi_1(X) \to \Gamma$ the analogous isomorphism, then $\psi \circ H^*(\alpha_Y) = H^*(\alpha_X)$; i.e., ψ commutes with the homomorphisms to $H^*(\Gamma, \mathbb{Z})$ induced by the classifying maps for α_Y, α_X respectively, or
- (b2) the same occurs for homology: there are isomorphisms $\phi_i : H_i(X, \mathbb{Z}) \to H_i(Y, \mathbb{Z})$ commuting with $H_i(\alpha_Y), H_i(\alpha_X)$, or
- (b') : there is an isomorphism $\alpha_Y : \pi_1(Y) \to \Gamma = \mathbb{Z}^{2n+2}$ such that, denoting by a_Y the corresponding classifying map,
- (b'1): $(a_Y)_*[Y]$ is dual to a polarization of type $\overline{d}$, and
- (b'2): $H_2(a_Y, \mathbb{Z})$ is an isomorphism.

Observe that Hypothesis (a) implies (a'), Hypothesis (b) implies (b1), (b1) implies (b2) by Poincaré duality, and (b2) implies (b'), (b'1) and (b'2).

Remark 0.10 *Assume Hypotheses (a'), (b'), (b'1) and (b'2) above.*

Then

I) the Albanese map $a_Y : Y \to Alb(Y) =: A'$ has image Σ which is an ample hypersurface, indeed $(a_Y)_*(Y) = deg(a_Y)\Sigma$ is the dual class of a polarization of type $(d_1, \dots, d_{n+1})$.

II) a) holds, i.e. Y is a projective manifold,

III) if $n \geq 3$, then a_Y is a finite map.

Remark 0.11 *When $n = dim(Y) = 2$ it can indeed happen that a_Y is not finite: since we may take Σ to be a hypersurface with Rational Double Points, and by Brieskorn-Tyurina's theorem, the minimal resolution of singularities Y is diffeomorphic to a smooth deformation X of Σ.*

The following characterization of smooth ample hypersurfaces in Abelian varieties is a refinement of a theorem obtained in [1]: in particular here the hypothesis that K_Y is ample is removed:

Theorem 0.12 *Assume that X is a smooth ample hypersurface in an Abelian variety, of dimension $n \geq 2$.*

Assume that Y is a compact Kähler manifold which satisfies the topological conditions (b'), (b'1) and (b'2) above.

Moreover, for $n \geq 3$, assume either:

(I) $K_Y^n = K_X^n = d_1 \dots d_{n+1}(n+1)!$, or

(II) $p_g(Y) = p_g(X) = d_1 \dots d_{n+1} + n$.

Whereas, for $n = 2$, assume either the topological condition (b1), or (I) above.

Denote the image of a_Y by W, and assume either that

(i) the class of X is indivisible (i.e., $d_1 = 1$), or the following consequence:

(ii) the degree of the map $a_Y : Y \to W$ equals 1

Then, for $n \geq 3$, the Albanese map a_Y yields an isomorphism:

$$a_Y : Y \cong W.$$

Whereas, for $n = 2$, a_Y is the minimal resolution of singularities of a canonical surface, i.e., a surface with Rational Double Points as singularities, and with ample canonical divisor.

Theorem 0.13 *Let $n \geq 2$ and $\overline{d} := (d_1, d_2, \dots, d_{n+1})$ be a polarization type for complex Abelian Varieties.*

Then, for $n \geq 3$, the smooth hypersurfaces of type $\overline{d}$ in some complex Abelian variety and the smooth Iterated Univariate Coverings of Normal Type and of type $\overline{d}$ form an irreducible connected component of the moduli space of canonically polarized manifolds, and also an irreducible connected component of the Kähler-Teichmüller space (of any such smooth hypersurface X).

For $n = 2$ we need also to include the minimal resolutions of such surfaces which have only Rational Double Points as singularities.

For $d_1 = 1$ there is only this connected component of the Kähler-Teichmüller space if

- *$n = 2$ or if*
- *we restrict ourselves to compact Kähler manifolds Y with $K_Y^n = (n+1)! d_1 d_2 \dots, d_{n+1}$, or if*
- *we restrict ourselves to compact Kähler manifolds Y with $p_g(Y) = d_1 d_2 \dots, d_{n+1} + n$.*

The Kähler-Teichmüller space is the open set of Teichmüller space corresponding to the Kähler complex structures (see [3] for general facts about Teichmüller space). In special situations (see [2]) it is a connected component of Teichmüller space.

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