A motivic formalism in representation theory

Shane KELLY (Tokyo Institute of Technology)*

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1. Representation theory

1.1. General background

In this talk, G is always a connected reductive linear algebraic group, equipped with a choice of Borel subgroup B and a split maximal torus T. We always work over an algebraically closed field $k = \overline{k}$, but we assume that G, B, T are defined over the integers $\mathbb{Z}$. In at least one equation, the two base changes $G(\mathbb{C})$ and $G(\overline{\mathbb{F}}_p)$ to $\mathbb{C}$ and $\overline{\mathbb{F}}_p$ will appear in the same formula.

Example 1. Examples of such groups are the following. Set $B = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ \iddots & & & & \end{pmatrix}$.

1. $T_n \cong \mathbb{G}_m^n = \{$ invertible diagonal $n \times n$ matricies $\}$,
2. $SL_n = \{$ $n \times n$ matricies M such that $\det M = 1\}$,
3. $Sp_{2n} = \{$ $2n \times 2n$ matricies M satisfying $M^t BM = B\}$
4. $SO_{2n+1} = \{$ $(2n+1) \times (2n+1)$ matricies M which satisfy both $M^t BM = B$ and $\det M = 1\}$

All of the above choices for G have a canonical choice of B and T:

$B = \{M \in G : M$ is upper triangular $\}$ and

$T = \{M \in G : M$ is diagonal $\}$.

One of the principle objects of study in representation theory are algebraic representations. That is, actions of G

$$G \times V \to V$$

on a finite dimensional k-vector space that are defined via polynomials.

Example 2. Given a *character*, i.e., an algebraic action $\lambda : T \times V \to V$ of T on the 1-dimensional k-vector space V, we can extend T to an action $B \times V \to V$ using the canonical retraction $B \to T$. Then the quotient of $G \times V$ by the diagonal action of B defines a vector bundle $\mathcal{O}(\lambda)$ on the variety of cosets G/B. The global sections of this vector bundle have a canonical algebraic G-action, induced by the action of G on the left component of $G \times V$. In this way, we obtain algebraic representations

$$\nabla(\lambda) = \Gamma(G/B, \mathcal{O}(\lambda)).$$

These are called *standard* representations.

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* e-mail: shanekelly@math.titech.ac.jp
web: http://www.math.titech.ac.jp/~shanekelly/indexjp.html

Example 3. In the case $G = SL_2$ there is a canonical isomorphism $G/B \cong \mathbb{P}^1$ sending $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in SL_2$ to $(a : c) \in \mathbb{P}^1$. The elements of $B \subseteq SL_2$ have the form $\left[\begin{smallmatrix} a & b \\ 0 & 1/a \end{smallmatrix}\right]$. For each $n \in \mathbb{Z}$ and vector space V, there is the B-action

$$\lambda_n : (\left[\begin{smallmatrix} a & b \\ 0 & 1/a \end{smallmatrix}\right], v) \mapsto a^n v.$$

If $\dim V = 1$, every action is of this form. Under the isomorphism $G/B \cong \mathbb{P}^1$, the vector bundle $\mathcal{O}(\lambda_n)$ on G/B corresponds to the standard line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1$. In particular, we have

$$\nabla(\lambda_n) = k[x, y]_n := \{\text{ homogeneous polynomials of degree } n\}$$

with the SL_2-action

$$(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], f(x, y)) \mapsto f(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]^{-1}(x, y)) = f(dx - by, ay - cx).$$

Remark 4. In the above example note that $\nabla(\lambda_n) = 0$ if $n < 0$. Characters λ such that $\nabla(\lambda) \neq 0$ are called *dominant*, [Jan07, Prop.2.6].

Remark 5. In the above example, if $\operatorname{char} k = p$ and $n > p$, then the representation $k[x, y]_n$ has the nonzero subrepresentation $k[x^p, y^p]_n$ consisting of those homogeneous polynomials of degree n whose terms are powers of p. For example, if $n = p$, then $\{\alpha_{p,0}x^p + \alpha_{0,p}y^p\} \subseteq \{\sum_{i+j=p} \alpha_{i,j}x^i y^j\}$ is G-invariant.

A representation $G \times V \to V$ is called *irreducible* if the only G-invariant subspaces $W \subseteq V$ are $\{0\}$ and V.

Theorem 6 ([Jan07, Cor.2.3, Cor.2.7]). *For every character $\lambda : T \times V \to V$, there is a unique nonzero irreducible subrepresentation*

$$L(\lambda) \subseteq \nabla(\lambda).$$

If $\operatorname{char} k = 0$, *then $L(\lambda) = \nabla(\lambda)$. In general,* every *irreducible representation is of the form $L(\lambda)$ for some λ.*

Every representation can be built from irreducible representations.

Theorem 7 (Jordan–Hölder). *If M is a representation, there exists a sequence $0 = M_{-1} \subsetneq M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$ such that M_i/M_{i-1} is irreducible. The set of* factors *M_i/M_{i-1} is independent of the chosen sequence.*

We write $[M : L(\lambda)]$ for the number of times the irreducible representation $L(\lambda)$ occurs as a subquotient of M.

Question A (Major goal of representation theory). Given two characters $\nu, \mu : T \times V \to V$, calculate the number $[\nabla(\nu) : L(\mu)]$ of times $L(\mu)$ appears as a subquotient of $\nabla(\nu)$.

1.2. Further remarks

Remark 8. In fact, both the set $\{[\nabla(\lambda)]\}$ of classes of standard representations, and the set $\{[L(\lambda)]\}$ of classes of irreducible representations form a basis for $K_0(Rep_G)$, the Grothendieck group of the category of representations.

$$\mathbb{Z}\Big\{[\nabla(\lambda)]\Big\} \cong K_0(Rep_G) \cong \mathbb{Z}\Big\{[L(\lambda)]\Big\} \tag{1}$$

With this in mind, Question A can be phrased as:

Question A′. Calculate the change of basis matrix of the isomorphism (1).

Remark 9. Given a representation $G \times V \to V$, we can restrict to a representation $T \times V \to V$. Since $T \cong \mathbb{G}_m^n$ (for some n), we have $K_0(Rep_T) \cong \mathbb{Z}[e_1, \ldots, e_n, e_1^{-1}, \ldots, e_n^{-1}]$. This is the polynomial algebra in n variables with all variables inverted, and e_i corresponds to the one dimensional representation $((a_1, \ldots, a_n), v) \mapsto a_i v$.

The *Weyl character formula* gives an expression for

$$\mathsf{im}([\nabla(\lambda)]) \in K_0(T) \cong \mathbb{Z}[e_1, \ldots, e_n, e_1^{-1}, \ldots, e_n^{-1}]$$

for any $\lambda : T \times V \to V$. This follows from an upper-triangularity argument involving the highest weights λ in $\nabla(\lambda)$. In fact the restriction map $K_0(Rep_G) \to K_0(Rep_T)$ is injective, hence another equivalent expression for Question A above is:

Question A″. Compute the image of $[L(\lambda)]$ under the restriction homomorphism $K_0(Rep_G) \to K_0(Rep_T) \cong \mathbb{Z}[e_1, \ldots, e_n, e_1^{-1}, \ldots, e_n^{-1}]$.

Remark 10. Note: these contain a lot of information. E.g., the modular characters of Sym_n are encoded in the $\mathsf{im}([L(\lambda)]) \in K_0(T)$ for $SL_n(\overline{\mathbb{F}}_p)$ (by a version of Schur-Weyl duality, and the Weyl character formula).

1.3. The modular category $\mathcal{O}$ (for $\operatorname{char} k > 0$)

The category Rep_G of representations suffers a number of problems. For example:

1. There are infinitely many (isomorphism classes of) irreducibles.

2. There are no nonzero projective objects.

Soergel introduced a modification of Rep_G which has nicer properties.

"Definition" 11. The *modular category* $\mathcal{O} \stackrel{def}{=} \mathcal{A}/\mathcal{N}$ is a "nice" subquotient of Rep_G.

That is, for a well-chosen pair of Serre abelian subcategories $0 \subseteq \mathcal{N} \subseteq \mathcal{A} \subseteq Rep_G$ that we will describe later, $\mathcal{O}$ is the quotient abelian category $\mathcal{A}/\mathcal{N}$. So objects of $\mathcal{O}$ are objects of $\mathcal{A}$, and morphisms of $\mathcal{O}$ are equivalence classes of "roofs" $x \stackrel{s}{\leftarrow} x' \to y$ of morphisms in $\mathcal{A}$, such that $\ker s \in \mathcal{N}, \operatorname{coker} s \in \mathcal{N}$. For more information about quotients of Serre abelian categories see [Stacks, Tag 02MN]. Note that there is a canonical functor

$$\mathcal{A} \to \mathcal{A}/\mathcal{N} = \mathcal{O}; \qquad M \mapsto \overline{M},$$

sending a representation to itself, but considered now as an object of $\mathcal{O}$.

The subcategories $\mathcal{A}, \mathcal{N}$ (which we will describe later) are chosen such that $\mathcal{O}$ has the following nice properties:

1. There are finitely many simple objects in $\mathcal{O}$. In fact, the simple objects of $\mathcal{O}$ are in canonical bijection with elements of the Weyl group $W = N_G(T)/T$. So, for example, in the case of $G = SL_n$, they are parametrised by the symmetric group Sym_n.

2. $\mathcal{O}$ has enough projectives. Moreover, for every character $\mu : T \times V \to V$ such that $L(\mu) \in \mathcal{A}$, there is a "smallest" epimorphism from a projective object in $\mathcal{O}$. We write
$$P(\mu) \to \overline{L(\mu)}$$

for this projective object $P(\mu) \in \mathcal{O}$. Moreover, these projective objects admit a flag $0 = F_{-1} \subsetneq F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = P(\mu)$ such that each subquotient (in $\mathcal{O}$) is isomorphic (in $\mathcal{O}$) to the image $\overline{\nabla(\nu)}$ of a standard representation. Finally, $[\nabla(\nu) : L(\mu)]$ is equal to the number $[P(\mu) : \overline{\nabla(\nu)}]$ of i such that $\overline{\nabla(\nu)} \cong F_i/F_{i-1}$.

$$\boxed{[\nabla(\nu) : L(\mu)] = [P(\mu) : \overline{\nabla(\nu)}].}$$

Note that if one can compute $[\nabla(\nu) : L(\mu)]$ for ν, μ such that $L(\mu), \nabla(\nu) \in \mathcal{A}$, a formula should be easily generalisable to all representations.

Remark 12. By BGG reciprocity, [Irv90],

$$\mathbb{Z}\Big\{[P(\mu)] : L(\mu) \in \mathcal{A}\Big\} \cong K_0(\mathcal{O}) \cong \mathbb{Z}\Big\{[\overline{L(\mu)}] : L(\mu) \in \mathcal{A}\Big\}. \tag{2}$$

That is, the projective objects $P(\mu)$ form a basis for the Grothendieck group of $\mathcal{O}$ (there is some redundency here, because $L(\mu) \in \mathcal{N}$ means $P(\mu), \overline{L(\mu)}$ is zero).

The complete definition of $\mathcal{A}$ and $\mathcal{N}$ involves the introduction of a lot of notation, but we give an idea about it here.

Idea 13. The idea for the definition of $\mathcal{O}$ is the following. There is a canonical *shifted p-dilated* action $\cdot_p : W \times X \to X$ (which we do not describe here) of the Weyl group $W = N_G(T)/T$ on the lattice of characters $X = \{\text{actions } T \times V \to V : \dim V = 1\}$. Moreover, there is a canonical orbit $W \cdot_p \xi = \{x \cdot_p \xi : x \in W\}$ for this action ($\xi \in X$ is some fixed canonical element). We *want* to define $\mathcal{A}$ as the full subcategory of representations whose irreducible subquotients lie in $W \cdot_p \xi$. However this is not a Serre abelian category, so we define

> $\mathcal{A}$:= the smallest full Serre abelian subcategory of Rep_G containing the irreducible representations $\{L(x \cdot_p \xi) : x \in W\}$.

The problem with $\mathcal{A}$ is that in addition to the irreducible representations $L(x \cdot_p \xi)$, it also contains other irreducible representations. This is where $\mathcal{N}$ comes in.

> $\mathcal{N}$:= the smallest full Serre abelian subcategory of $\mathcal{A}$ containing the irreducible representations *not* in $\{L(x \cdot_p \xi) : x \in W\}$.

The properties of $\mathcal{O}$ are more important to us than the precise definition, so we direct the reader to [Jan07] for the precise definition of $\cdot_p$ and ξ (there ξ is denoted $st + \lambda$).

1.4. Perverse sheaves

Now we move the focus to geometry. As an algebraic variety, the variety of cosets G/B is particularly simple. It has a canonical decomposition into locally closed subvarieties

$$G/B = \bigcup_{x \in W} BxB/B$$

indexed by the Weyl group $W = N_G(T)/T$. Moreover, for each $x \in W$ there is an isomorphism $BxB/B \cong \mathbb{A}^{n_x}$ for some n_x. For example, in the case $G = SL_2$, this is the canonical decomposition $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$. However, the combinatorics of how these strata sit together contains a lot of information about the representation theory of G. As such, the geometry of G/B can be used in a very strong way to study the representation theory of G.

For example, Soergel's proof of the Kazhdan-Lusztig conjecture (for $k = \mathbb{C}$) uses an isomorphism

$$\bigoplus_i \hom_{D^b((G/B)(\mathbb{C}),\mathbb{C})}(IC_x, IC_y[i]) \cong \hom_{\mathcal{O}_0}(P(x), P(y)). \tag{3}$$

Here, $(G/B)(\mathbb{C})$ is the variety of complex points of G/B considered as a stratified space, $D^b((G/B)(\mathbb{C}), \mathbb{C})$ is the category of $\mathbb{C}$-valued sheaves on $(G/B)(\mathbb{C})$, the object IC_x is the intersection complex associated to the closure $\overline{BxB/B} \subseteq G/B$ of the stratum BxB/B, $x, y \in W$, and $\mathcal{O}_0, P(x), P(y)$ are the complex Lie algebra inspirations for $\mathcal{O}$ and $P(\mu)$ from above.

Remark 14. The isomorphism (3) uses the *decomposition theorem* which says that for any smooth proper map $f : X \to Y$ between algebraic varieties, the derived push-forward $Rf_*\mathbb{C}$ of the constant sheaf on X decomposes in $D^b(Y, \mathbb{C})$ into the sum of its cohomology sheaves $Rf_*\mathbb{C} \cong \oplus_{i=-d}^{d} R^{d+i} f_*\mathbb{Q}[-d-i]$.

Remark 15. If $\operatorname{char} k = p$ and we use sheaves of $\overline{\mathbb{F}}_p$-vector spaces instead of sheaves of $\mathbb{C}$-vector spaces, the decomposition theorem is *false*!

Soergel's approach to the positive characteristic Lusztig's conjecture is to replace IC_x with *parity sheaves.* A complex $E \in D^b((G/B)(\mathbb{C}), \overline{\mathbb{F}}_p)$ is *even* if its cohomology on each stratum vanishes in odd degrees. It is *odd* if $E[1]$ is even. It is *parity* if $E = E_0 \oplus E_1$ for some even complex E_0 and odd complex E_1. See [Wil18] for more details. Using parity sheaves in place of intersection complexes, one obtains an $\overline{\mathbb{F}}_p$-linear version of (3).

$$\bigoplus_i \hom_{D^b((G/B)(\mathbb{C}),\overline{\mathbb{F}}_p)}(E_x, E_y[i]) \cong \hom_{\mathcal{O}}(P(\mu_x), P(\mu_y)) \tag{4}$$

Here, $P(\mu)$ are in the modular category $\mathcal{O}$ described above.

A natural question is the following:

Question 16. Can we upgrade (4) to a sheaf description of all of $\mathcal{O}$?

2. Motivic sheaves

2.1. Motives

Sheaves and sheaf cohomology on algebraic varieties often comes equipped with extra structure that is extremely useful. In fact, D-modules and perverse sheaves are the backbone of geometric representation theory. For example, D-modules and perverse sheaves are used in Beilinson–Bernstein's, Brylinski–Kashiwara's proof of the Kazhdan-Lusztig conjecture, as well as many other places in representation theory. Similarly, in the Weil conjectures which motivated the development of étale cohomology, the action of the absolute Galois group of the base field on the étale cohomology plays a vital rôle.

Indeed, there is a strong analogy between Hodge structures and Galois representations. In order to make this analogy concrete, Grothendieck introduced the notion of motives. Loosely speaking, a *motive* is an object which contains information about the Hodge theory, the Galois representations, and even the F-crystal associated to a variety (see [Del89] for a definition of motives in terms of this data). Moreover, just as families of Hodge structures and Galois representations are expressed in terms of sheaves, there is also a relative theory of motives.

First let us describe the rational coefficients theory which is a little simpler. In this case, one can ([Ayo14], [CD19], [Rob12], [Voe00]) define the category of motives over a base variety S as the universal functor

$$M : \mathsf{Sm}/S \to DA(S)$$

towards a $\mathbb{Q}$-linear tensor triangulated category[1] satisfying:

1. ($\mathbb{A}^1$-invariance) $M(\mathbb{A}^1 \times S) \cong M(S)$. Heuristically, the cohomology of $\mathbb{A}^1$ should be trivial.

2. (Excision) If $U \to X$ is an open morphism, and $f : V \to X$ an étale morphisms such that $f^{-1}(X \setminus U) \cong X \setminus U$, then there is a distinguished triangle $M(U \times_X V) \to M(U) \oplus M(V) \to M(X) \xrightarrow{+}$. Heuristically, if $V \to X$ is a "tubular" neighbourhood of $Z = X \setminus U$, then the relative cohomology of $(V, (V \setminus Z))$ is the same as the relative cohomology of (X, U).

3. (Stability) $Ker(M(\mathbb{P}^1_S) \to M(S))$ is tensor invertible. Heuristically, the second cohomology vector space of $\mathbb{P}^1$ should be one dimensional. One writes $\mathbb{Q}(1)$ for $Ker(M(\mathbb{P}^1_S) \to M(S))[-2]$ and $\mathbb{Q}(n)$ for its n-fold tensor powers, $n \in \mathbb{Z}$.

4. (Galois descent) For any finite étale Galois cover Y/X with Galois group G, we have $M(Y)^G \cong M(X)$.

Remark 17. One can think about objects of $DA(S)$ as something resembling families of Hodge structures or Galois representations, and the functor M sends a smooth S-variety $f : X \to S$ to $Rf_!\mathbb{Q}$. Indeed, if $\ell \neq \operatorname{char} k$, by the universal property in the definition we gave, there is a canonical tensor triangulated functor

$$DA(S) \to D_{et}(S, \mathbb{Q}_\ell)$$

towards the ℓ-adic category of S sending $M(X \xrightarrow{f} S)$ to $Rf_!\mathbb{Q}_\ell$. If $\ell = \operatorname{char} k$, and we use $\overline{\mathbb{F}}_p$-linear categories instead of $\mathbb{Q}$-linear, then of course the étale cohomology of $\mathbb{A}^1$ is nontrivial, and there is no such functor. If $\operatorname{char} k = 0$ there is also the analogous de Rham realisation functor, cf. [Ayo15].

Amazingly, one can deduce very strong properties from just these four axioms. Similar to ℓ-adic cohomology, for every morphism $f : X \to Y$ of varieties, we have the following adjunctions, where $\mathcal{E}$ is any object of $DA(X)$. We write f_* instead of Rf_* etc, because from now on we only deal with "derived" functors.

$$f^* : DA(Y) \rightleftarrows DA(X) : f_*$$

$$f_! : DA(X) \rightleftarrows DA(Y) : f^!$$

$$- \otimes \mathcal{E} : DA(X) \rightleftarrows DA(X) : \mathcal{H}om(\mathcal{E}, -)$$

[1] Strictly speaking, one must use infinity categories for this style of definition.

These adjunctions satisfy a number of nice properties such as smooth base change, proper base change, purity, Verdier duality, a projection formula, ..., [Ayo07]. Furthermore, just as

$$\hom_{D_{et}(X,\mathbb{Q}_\ell)}(\mathbb{Q}_\ell, \mathbb{Q}_\ell(i)[j]) \cong H^j_{et}(X, \mathbb{Q}_\ell(i))$$

in the étale theory, we have, [Kel17], [Voe00],

$$\hom_{DA(S)}(\mathbb{Q}, \mathbb{Q}(i)[j]) \cong CH^i(X, 2i{-}j; \mathbb{Q}),$$

where the groups on the right are Bloch's higher Chow groups, [Blo86], with $\mathbb{Q}$-coefficients, $\mathbb{Q} = M(S)$ is the tensor unit, and $\mathbb{Q}(i)[j]$ is the jth shift of the ith tensor power of $Ker(M(\mathbb{P}^1_S) \to M(S))[-2]$.

When $\operatorname{char} k = p > 0$ and we use $\overline{\mathbb{F}}_p$-linear categories, one considers the universal category $D_{\mathbb{A}^1}(S, \overline{\mathbb{F}}_p)$ as defined above, but only satisfying axioms (1) $\sim$ (3).[2] The category $D_{\mathbb{A}^1}(S, \overline{\mathbb{F}}_p)$ does not have quite the right hom groups for our representation theoretic purposes. However, there is a monoid object (the object representing Milnor K-theory with $\overline{\mathbb{F}}_p$-coefficients)

$$\mathbb{K} \in D_{\mathbb{A}^1}(S, \overline{\mathbb{F}}_p)$$

such that the category of $\mathbb{K}$-modules

$$H(S) := \mathbb{K}{-}\mathsf{mod}$$

has the right hom groups. Indeed, leveraging work of Geisser-Levine [GL00] we have[3]

$$\hom_{H(S)}(\overline{\mathbb{F}}_p, \overline{\mathbb{F}}_p(i)[j]) = CH^i(S, 2i{-}j; \overline{\mathbb{F}}_p).$$

Specifically, the property we need is:

$$\hom_{H(\mathbb{A}^n_{\overline{\mathbb{F}}_p})}(\overline{\mathbb{F}}_p, \overline{\mathbb{F}}_p(i)[j]) = \begin{cases} \overline{\mathbb{F}}_p & (i,j) = (0,0) \\ 0 & (i,j) \neq (0,0) \end{cases} \tag{5}$$

Again, we have the six functors

$$f^* : H(Y) \rightleftarrows H(X) : f_*$$

$$f_! : H(X) \rightleftarrows H(Y) : f^!$$

$$- \otimes \mathcal{E} : H(X) \rightleftarrows H(X) : \mathcal{H}om(\mathcal{E}, -)$$

satisfying smooth base change, proper base change, purity, Verdier duality, a projection formula, ..., [Ayo07], [CD19].

Remark 18. Again, strictly speaking, one must use infinity categories if one wants to use these specific definitions. Indeed, for a general monoid M in a tensor triangulated category the category of M-modules is rarely triangulated. If the reader does not enjoy infinity categories, there is a perfectly concrete construction of $\mathbb{K}{-}\mathsf{mod}$ as a Verdier quotient of the derived category of an abelian category. I.e., a construction which uses only homological algebra from 1967. See [EK19] for this construction explained in detail.

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[2] Indeed, if one asks for Galois descent in the presence of $\mathbb{A}^1$-invariance, then by the Artin-Schreier sequence, one obtains the zero category.

[3] This isomorphism holds when S is smooth; one would have this formula in general if resolution of singularities was known in positive characteristic.

Remark 19. The category $H(S) = \mathbb{K}-\mathsf{mod}$ is essentially modules over the motivic Eilenberg-Maclane spectrum with $\overline{\mathbb{F}}_p$-coefficients $H\overline{\mathbb{F}}_p$. Indeed, if resolution of singularities holds over $\mathbb{F}_p$, it can be shown that $H(S)$ is equivalent to Voevodsky's category of motives $DM(S, \overline{\mathbb{F}}_p)$. However, we wanted to make the category as accessible as possible to non-algebraic geometers, and so we choose to use Milnor K-theory instead in order to avoid any discussion of correspondances or cycles.

2.2. Mixed Tate motives

The category of all motives is too big, so now we restrict our attention to motives which are "locally constant along a stratification".

Definition 20. An *affinely stratified variety* is a variety X with a finite partition $\mathcal{S}$ into locally closed subvarieties (called the strata of X)

$$X = \bigcup_{s \in \mathcal{S}} X_s$$

such that each stratum X_s is isomorphic to $\mathbb{A}^n$ for some n, and the closure $\overline{X}_s$ is a union of strata. The embeddings are denoted by $j_s : X_s \hookrightarrow X$.

Example 21. The motivating example we have in mind is G/B with the stratification $G/B = \cup_{x \in W} BxB/B$. For SL_2 this is $\mathbb{P}^1 = \mathbb{A}^1 \amalg \{\infty\}$. For SL_n it is the flag variety.

Definition 22. The category of *mixed Tate motives* on X, is the smallest full triangulated subcategory of $H(X)$ containing the motives $\overline{\mathbb{F}}_p(i)[j]$.

$$\mathrm{MTDer}(X) = \langle \overline{\mathbb{F}}_p(n)[j] \rangle \subset \mathsf{H}(X, \overline{\mathbb{F}}_p).$$

and the category of *stratified mixed Tate motives* on $(X, \mathcal{S})$ is

$$\mathrm{MTDer}_{\mathcal{S}}(X) = \{M \in \mathsf{H}(X, \overline{\mathbb{F}}_p) : j_s^* M \in \mathrm{MTDer}(X_s) \ \forall\ s \in \mathcal{S}\}.$$

In other words, the category of motives which are mixed Tate along each X_s.

Remark 23. Notice that if we ran the above definition with the category $D_{et}(X, \mathbb{Q}_\ell)$ instead of $H(X)$, we obtain complexes of constructible sheaves.

Remark 24. If we consider $\mathbb{A}^n$ as an affinely stratified variety with a single stratum, then

$$\mathrm{MTDer}_{\mathcal{S}}(\mathbb{A}^n) = \mathrm{MTDer}(\mathbb{A}^n).$$

Moreover, by the vanishing (5) mentioned above, there is a canonical equivalence between $\mathrm{MTDer}_{\mathcal{S}}(\mathbb{A}^n)$ and the bounded derived category of graded $\overline{\mathbb{F}}_p$-vector spaces,

$$\mathrm{MTDer}_{\mathcal{S}}(\mathbb{A}^n) \cong D^b(\overline{\mathbb{F}}_p\text{-vec.sp.}^{\mathbb{Z}}).$$

This equivalence sends $\overline{\mathbb{F}}_p(i)$ to the one dimensional vector space of weight i. In particular, it makes sense to talk about the *dimension* of objects of $\mathrm{MTDer}_{\mathcal{S}}(\mathbb{A}^n)$.

The subcategory of stratified mixed Tate motives is of great interest in representation theory. Using Soergel's results one can prove the following.

Theorem 25 ([EK19])**.** *Let G be a semisimple simply connected split algebraic group over $\overline{\mathbb{F}}_p$ and $G^\vee$ the Langlands dual group. Then there is an equivalence of categories*

$$\mathrm{MTDer}_{(B^\vee)}(G^\vee/B^\vee) \xrightarrow{\sim} \mathrm{Der}^{\mathrm{b}}(\mathcal{O}^{2\mathbb{Z}}(G))$$

between the category of stratified mixed Tate motives on $G^\vee/B^\vee$ and the derived evenly graded modular category $\mathcal{O}^{2\mathbb{Z}}(G)$.

Remark 26. We have to assume that both the torsion index of G is invertible in $\mathbb{F}_p$ and p is bigger than the Coxeter number of G.

Remark 27. The category $\mathcal{O}^{2\mathbb{Z}}(G)$ is a graded version of the category $\mathcal{O}$ we discussed above.

2.3. Application

Now we come back to the numbers $[\Delta(\nu) : L(\mu)]$ that we were interested in in the beginning. We begin with the following.

Proposition 28. *For all morphisms $f : X \to X'$ of interest to us, the adjunctions*

$$(f_!, f^!), \qquad (f^*, f_*), \qquad (\otimes, \mathcal{H}om)$$

restrict to $\mathrm{MTDer}_{\mathcal{S}}(X), \mathrm{MTDer}_{\mathcal{S}'}(X')$.

Some examples of such morphisms that we have in mind are the following.

Example 29. Consider $(SL_n/B, \mathcal{B})$ where $\mathcal{B}$ is the Bruhat stratification. For all $s \in \mathcal{B}$, we have the "skyscraper sheaf"

$$j_{s!}\overline{\mathbb{F}}_p \in \mathrm{MTDer}_{\mathcal{B}}(SL_n/B).$$

Example 30. For any parabolic subgroup $B \subseteq P \subseteq SL_n$ we have the projection

$$\pi : SL_n/B \to SL_n/P.$$

Then we have an endofunctor

$$\pi^*\pi_* : \mathrm{MTDer}_{\mathcal{B}}(SL_n/B) \to \mathrm{MTDer}_{\mathcal{B}}(SL_n/B).$$

Let us now combine the two examples above.

Example 31. Let s be a simple reflection in the Weyl group $W = N_G T/T \cong Sym_n$ of SL_n. Let P_s be the parabolic subgroup associated to s, consider the $\mathbb{P}^1$-bundle $\pi_s : G/B \to G/P_s$, and consider its associated endofunctor

$$\Pi_s = \pi_s^*\pi_{s*} : \mathrm{MTDer}_{\mathcal{B}}(SL_n/B) \to \mathrm{MTDer}_{\mathcal{B}}(SL_n/B).$$

For any decomposition $w = s_1 \dots s_n$ of an element $x \in W$ of the Weyl group into a product of simple reflections, we obtain an endofunctor

$$\Pi_x = \Pi_{s_1} \circ \cdots \circ \Pi_{s_n} : \mathrm{MTDer}_{\mathcal{B}}(SL_n/B) \to \mathrm{MTDer}_{\mathcal{B}}(SL_n/B).$$

Let pt be the stratum $B/B \subseteq SL_n/B$, and consider the "skyscraper" motive

$$\mathcal{P} = pt_!\overline{\mathbb{F}}_p.$$

The object $\Pi_x\mathcal{P}$ has a unique indecomposable direct summand $\mathcal{P}_x$ with support on the Bruhat cell of SL_n/B corresponding to x, and for any other element $y \in W$, we have:

$$\boxed{\dim j_y^*\mathcal{P}_x = [\nabla(\nu) : L(\mu)].}$$

where $\xi : T \times V \to V$ is the character inducing the bijection between elements of the Weyl group and simple objects of $\mathcal{O}$, and $\mu = \xi \cdot_p x, \nu = \xi \cdot_p y$ are the characters corresponding to $x, y \in W$. See Remark 24 for the notion of dimension.

In other words, the category of motives $\mathrm{MTDer}_{\mathcal{B}}(SL_n/B)$ contains information about the modular representation theory of Sym_n.

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