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Cichoń's maximum over ZFC alone

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1. Introduction

Cichoń's diagram (Figure 1) illustrates the relationship between the classical cardinal characteristics of the continuum associated with Baire category, Lebesgue measure and compactness of subsets of the irrationals. To describe it in full, we first introduce some notation.

Given $\mathcal{I} \subseteq \mathcal{P}(\mathbb{R})$, denote

$$\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I},\ \bigcup\mathcal{A} \notin \mathcal{I}\};$$

$$\mathrm{cov}(\mathcal{I}) = \min\{|\mathcal{C}| : \mathcal{C} \subseteq \mathcal{I},\ \bigcup\mathcal{C} = \mathbb{R}\};$$

$$\mathrm{non}(\mathcal{I}) = \min\{|F| : F \subseteq \mathbb{R},\ F \notin \mathcal{I}\};$$

$$\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{C}| : \mathcal{C} \subseteq \mathcal{I},\ (\forall X \in \mathcal{I})(\exists Y \in \mathcal{C})\, X \subseteq Y\}.$$

Let $\mathcal{N}$ be the family of Lebesgue measure zero subsets of $\mathbb{R}$, $\mathcal{M}$ the family of meager subsets of $\mathbb{R}$, and let $\mathcal{K}$ be the family of subsets of $\mathbb{R}$ whose intersection with the irrationals is contained in some σ-compact subset of the irrationals. It is known that $\mathrm{add}(\mathcal{K}) = \mathrm{non}(\mathcal{K})$ and $\mathrm{cov}(\mathcal{K}) = \mathrm{cof}(\mathcal{K})$, so they are denoted by $\mathfrak{b}$ and $\mathfrak{d}$ respectively (the standard definition of $\mathfrak{b}$ and $\mathfrak{d}$ is presented in Example 2.2(4)). Denote $\mathfrak{c} := 2^{\aleph_0} = |\mathbb{R}|$, and recall that $\aleph_1$ is the smallest uncountable cardinal. These cardinals describe the entries in Cichoń's diagram (Figure 1).

$$\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & \mathfrak{c} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

Figure 1: Cichoń's diagram. An arrow indicates that ZFC proves $\leq$ between both cardinals. In addition, $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$.

Since the decade of 1980's Cichoń's diagram has been a relevant object of research in set theory of the reals, in particular linked with forcing theory. It has been proved that this diagram is complete, in the sense that no other inequalities (consistent with the diagram) can be proved in ZFC. See [1] for a complete survey about this diagram and its completeness.

One of the main questions is whether it is consistent that all entries in Cichoń's diagram (with the exception of the dependant values $\mathrm{add}(\mathcal{M})$ and $\mathrm{cof}(\mathcal{M})$) are pairwise

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different. One of the first results in this direction is due to Brendle [3], who showed examples of forcing extensions with finite support iterations where, with the exception of $\aleph_1$, all the entries can assume two arbitrary (regular) values. His techniques triggered relevant advances in this direction, like models with matrix (finite support) iterations where the diagram is separated into 6 different values [13], and afterwards into 7 different values [5, 14]. Also in the context of finite support iterations, there is a model where all cardinals of the left side of the diagram are pairwise different [7] (6 values), which was recently improved in [4] by separating one additional value on the right side (another model of 7 values).

In the previous examples, the right side of the diagram is separated into at most three different values. On the other hand, with large products of creature forcing, it is possible to separate the right side into 5 values [6]. However, forcing posets constructed in this way are ω^ω-bounding, hence they force $\mathfrak{d} = \aleph_1$. This is a limitation of this method to separate the left side of the diagram at the same time.

Very recently, assuming the existence of four strongly compact cardinals, it was proved one example where Cichoń's diagram assumes 10 different values [10], which is the maximum number of possible different values. To do this, the forcing construction in [7] to separate the left side was improved, and then the method of Boolean ultrapowers [12, 10] was used to separate the right side in addition. The same method of Boolean ultrapowers applied to the forcing in [4] allowed to reduce the large cardinal hypothesis to three strongly compact cardinals. Another example of 10 values, also using four strongly compact cardinals and Boolean ultrapowers, was presented in [11] (see also [15] for further remarks).

Just in this year, M. Goldstern, J. Kellner, S. Shelah and the author [9] proved that a model of Cichoń's diagram with 10 different values (as in [10]) can be obtained without any use of large cardinals. This method also starts with a ccc forcing $\mathbb{P}^-$ separating the left side, but the Boolean ultrapower method is replaced by intersecting the forcing with σ-closed elementary submodels of H_χ (for some large enough regular χ). Depending on the structure of such submodels, the shape of some "strong witnesses" of the cardinal characteristics added by $\mathbb{P}^-$ can be modified so that the right side is forced to be separated by the forcing resulting after the intersection of $\mathbb{P}^-$ with the submodels. The same method also allows to force the same example from [11] without using large cardinals. Even more, this method can be combined with techniques from [8] to separate other cardinal characteristics of the continuum in addition to those in Cichoń's diagram, concretely, $\aleph_1 < \mathfrak{m} < \mathfrak{p} < \mathfrak{h} < \mathrm{add}(\mathcal{N})$ can be forced in addition.

In the following sections, we reproduce the proof of the main result of [9], stated in Theorem 4.2. There are slight variations in this presentation, mostly associated with the author's style and perspective.

2. Relational systems

Many cardinal characteristics can be described in the following general context, see e.g. [2]

Definition 2.1. Say that $\mathbf{R} = \langle X, Y, \sqsubset \rangle$ is a *relational system* if $\sqsubset$ is a relation contained in $X \times Y$.

(1) A set $F \subseteq X$ is $\mathbf{R}$-*bounded* if $(\exists y \in Y)(\forall x \in F)\, x \sqsubset y$. Otherwise, it is $\mathbf{R}$-*unbounded.*

(2) A set $D \subseteq Y$ is $\mathbf{R}$-*dominating* if $(\forall x \in X)(\exists y \in D)\, x \sqsubset y$.

These notions allow to define the following cardinal characteristics:

$$\mathfrak{b}(\mathbf{R}) := \min\{|F| : F \subseteq X \text{ is } \mathbf{R}\text{-unbounded}\},$$

$$\mathfrak{d}(\mathbf{R}) := \min\{|D| : D \subseteq Y \text{ is } \mathbf{R}\text{-dominating}\}.$$

Denote $\mathbf{R}^\perp := \langle Y, X, \not\sqsupset\rangle$ the *dual of* $\mathbf{R}$.

It is clear that any $F \subseteq X$ is $\mathbf{R}$-unbounded iff it is $\mathbf{R}^\perp$-dominating, and $D \subseteq Y$ is $\mathbf{R}$-dominating iff it is $\mathbf{R}^\perp$-unbounded.

Inequalities between cardinal invariants are often proved using the *Tukey order* between relational systems. If $\mathbf{R} = \langle X, Y, \sqsubset\rangle$ and $\mathbf{R}' = \langle X', Y', \sqsubset'\rangle$ are relational systems, $\mathbf{R} \preceq_{\mathrm{T}} \mathbf{R}'$ means that there are two maps $\varphi : X \to X'$ and $\psi : Y' \to Y$ such that, for any $x \in X$ and $y' \in Y'$, $\varphi(x) \sqsubset' y'$ implies $x \sqsubset \psi(y')$. In this case, the ψ-image of any $\mathbf{R}'$-dominating set is $\mathbf{R}$-dominating, and the φ-image of any $\mathbf{R}$-unbounded set is $\mathbf{R}'$-unbounded, thus $\mathfrak{b}(\mathbf{R}') \leq \mathfrak{b}(\mathbf{R})$ and $\mathfrak{d}(\mathbf{R}) \leq \mathfrak{d}(\mathbf{R}')$. Say that $\mathbf{R}$ and $\mathbf{R}'$ are *Tukey equivalent*, denoted by $\mathbf{R} \cong_{\mathrm{T}} \mathbf{R}'$, if $\mathbf{R} \preceq_{\mathrm{T}} \mathbf{R}'$ and $\mathbf{R}' \preceq_{\mathrm{T}} \mathbf{R}$.

In the following example we show how the entries in Cichoń's diagram can be defined through very simple relational systems (compare with Definition 3.1).

Example 2.2. We fix some notation. For any fuction $b : \omega \to V \setminus \{\emptyset\}$ and $h \in \omega^\omega$, denote $\prod b := \prod_{i<\omega} b(i)$ and $\mathcal{S}(b, h) := \prod_{i<\omega} [b(i)]^{\leq h(i)}$. For two functions x, y with domain ω, $x \in^* y$ denotes $(\exists n \in \omega)(\forall i \geq n)\, x(i) \in y(i)$. Denote by id_ω the identity function on ω.

(1) Denote $\mathbf{Id} := \langle 2^\omega, 2^\omega, =\rangle$. Note that $\mathfrak{b}(\mathbf{Id}) = 2$ and $\mathfrak{d}(\mathbf{Id}) = \mathfrak{c}$.

(2) For $\mathcal{H} \subseteq \omega^\omega$ denote $\mathbf{Lc}(\omega, \mathcal{H}) := \langle \omega^\omega, \left([\omega]^{<\omega}\right)^\omega, \in^*_{\mathcal{H}}\rangle$ where $x \in^*_{\mathcal{H}} \varphi$ iff $x \in^* \varphi$ and $(\exists h \in \mathcal{H})\, \varphi \in \mathcal{S}(\omega, h)$ (here, ω denotes the constant function with value ω). If $\mathcal{H}$ is countable and contains a function that goes to infinity, then $\mathfrak{b}(\mathbf{Lc}(\omega, \mathcal{H})) = \mathrm{add}(\mathcal{N})$ and $\mathfrak{d}(\mathbf{Lc}(\omega, \mathcal{H})) = \mathrm{cof}(\mathcal{N})$ (see [1, Thm. 2.3.9]). For the rest of this paper, fix $\mathcal{H}_* := \{\mathrm{id}_\omega^{n+1} : n < \omega\}$ and $\mathbf{Lc} := \mathbf{Lc}(\omega, \mathcal{H}_*)$.

(3) Define $\Omega_n := \{a \in [2^{<\omega}]^{<\aleph_0} : \mathrm{Lb}(\bigcup_{s\in a}[s]) \leq 2^{-n}\}$ (endowed with the discrete topology), where Lb denotes the Lebesgue measure in the Cantor space 2^ω. Set $\Omega := \prod_{n<\omega} \Omega_n$ with the product topology, which is a perfect Polish space. For every $x \in \Omega$ denote $N_x^* := \bigcap_{n<\omega} \bigcup_{s\in x(n)} [s]$, which is clearly a Borel null set in 2^ω.

Define $\mathbf{Cn} := \langle \Omega, 2^\omega, \sqsubset\rangle$ where $x \sqsubset z$ iff $z \notin N_x^*$. Recall that any null set in 2^ω is a subset of N_x^* for some $x \in \Omega$, so $\mathbf{Cn} \cong_{\mathrm{T}} \langle \mathcal{N}(2^\omega), 2^\omega, \not\ni\rangle$. Hence, $\mathfrak{b}(\mathbf{Cn}) = \mathrm{cov}(\mathcal{N})$ and $\mathfrak{d}(\mathbf{Cn}) = \mathrm{non}(\mathcal{N})$.

(4) For $x, y \in \omega^\omega$ denote $x \leq^* y$ iff $(\exists n \in \omega)(\forall i \geq n)\, x(i) \leq y(i)$. The relational system $\mathbf{D} = \langle \omega^\omega, \omega^\omega, \leq^*\rangle$ describes the *unbounding number* $\mathfrak{b} = \mathfrak{b}(\mathbf{D})$ and the *dominating number* $\mathfrak{d} = \mathfrak{d}(\mathbf{D})$.

(5) Denote $\Xi := \{f : 2^{<\omega} \to 2^{<\omega} : \forall s \in 2^{<\omega}(s \subseteq f(s))\}$ and set $\mathbf{Mg} := \langle 2^\omega, \Xi, \in^\bullet\rangle$ where $x \in^\bullet f$ iff $|\{s \in 2^{<\omega} : x \supseteq f(s)\}| < \aleph_0$. Since $\mathbf{Mg} \cong_{\mathrm{T}} \langle 2^\omega, \mathcal{M}, \in\rangle$, $\mathfrak{b}(\mathbf{Mg}) = \mathrm{non}(\mathcal{M})$ and $\mathfrak{d}(\mathbf{Mg}) = \mathrm{cov}(\mathcal{M})$.

Below, we define very special types of dominating (and unbounded) families for relational systems. These play an important role when forcing values to cardinal characteristics.

Definition 2.3. Fix a relational system $\mathbf{R} = \langle X, Y, \sqsubset \rangle$ and a cardinal θ.

(1) For a set M say that $y \in Y$ is $\mathbf{R}$-*dominating over* M if $x \sqsubset y$ for all $x \in X \cap M$.

(2) Say that $x \in X$ is $\mathbf{R}$-*unbounded over* M if it is $\mathbf{R}^\perp$-dominating over M, that is, $x \not\sqsubset y$ for all $y \in Y \cap M$.

(3) A family $D \subseteq Y$ is θ-$\mathbf{R}$-*dominating* if, for every $E \in [X]^{<\theta}$, there is some $\mathbf{R}$-dominating $y \in D$ over E.

(4) A family $F \subseteq X$ is θ-$\mathbf{R}$-*unbounded* if it is θ-$\mathbf{R}^\perp$-dominating, that is, for any $H \in [Y]^{<\theta}$ there is some $\mathbf{R}$-unbounded $x \in F$ over H.

(5) A family $D \subseteq Y$ is *strongly* θ-$\mathbf{R}$-*dominating* if $|D| \geq \theta$ and, for every $x \in X$, $|\{y \in D : x \not\sqsubset y\}| < \theta$.

(6) A family $F \subseteq X$ is *strongly* θ-$\mathbf{R}$-*unbounded* if it is strongly-θ-$\mathbf{R}^\perp$-dominating, that is, $|F| \geq \theta$ and, for every $y \in Y$, $|\{x \in F : x \sqsubset y\}| < \theta$.

Remark 2.4. Fix a relational system $\mathbf{R} = \langle X, Y, \sqsubset \rangle$.

(1) The cardinal invariants $\mathfrak{b}(\mathbf{R})$ and $\mathfrak{d}(\mathbf{R})$ may not always exist. Concretely, $\mathfrak{b}(\mathbf{R})$ does not exist iff $\mathfrak{d}(\mathbf{R}) = 1$. Dually, $\mathfrak{d}(\mathbf{R})$ does not exists iff $\mathfrak{b}(\mathbf{R}) = 1$.

(2) Any subset of Y is $\mathbf{R}$-dominating iff it is 2-$\mathbf{R}$-dominating. Likewise, $\mathbf{R}$-unbounded is equivalent to 2-$\mathbf{R}$-unbounded.

(3) If $\theta \leq \theta'$ are cardinals, then any θ'-$\mathbf{R}$-dominating family is θ-$\mathbf{R}$-dominating. Likewise for unbounded families.

(4) If $\theta \geq 2$, then any θ-$\mathbf{R}$-dominating family is $\mathbf{R}$-dominating.[1] Likewise for unbounded.

(5) Any strongly θ-$\mathbf{R}$-dominating family is $\mathbf{R}$-dominating. Likewise for unbounded.

The following result shows the effect of these special types of dominating (and unbounded) families on cardinal characteristics.

Lemma 2.5. *Let $\mathbf{R} = \langle X, Y, \sqsubset \rangle$ be a relational system and $\theta \geq 2$ a cardinal.*

(a) If $D \subseteq Y$ is a θ-$\mathbf{R}$-dominating family then $\mathfrak{d}(\mathbf{R}) \leq |D|$ and $\theta \leq \mathfrak{b}(\mathbf{R})$

(b) If $F \subseteq X$ is θ-$\mathbf{R}$-unbounded then $\mathfrak{b}(\mathbf{R}) \leq |F|$ and $\theta \leq \mathfrak{d}(\mathbf{R})$.

(c) If θ is regular and $D \subseteq Y$ is a strongly θ-$\mathbf{R}$-dominating family, then D is $|D|$-$\mathbf{R}$-dominating, in particular, $\mathfrak{d}(\mathbf{R}) \leq |D| \leq \mathfrak{b}(\mathbf{R})$.

(d) If θ is regular and $F \subseteq X$ is strongly θ-$\mathbf{R}$-unbounded then it is $|F|$-$\mathbf{R}$-unbounded and $\mathfrak{b}(\mathbf{R}) \leq |F| \leq \mathfrak{d}(\mathbf{R})$.

We finish this section with the following special type of dominating family. This is extracted from the property COB (see Definition 3.2(2)), originally defined in [12, 10].

[1] Any subset of Y is 0-$\mathbf{R}$-dominating; and $D \subseteq Y$ is 1-$\mathbf{R}$-dominating iff $D \neq \emptyset$.

Definition 2.6. Let $\mathbf{R} = \langle X, Y, \sqsubset \rangle$ be a relational system and let $\langle S, \leq \rangle$ be a directed partial order. Say that $D \subseteq Y$ is *strongly S-$\mathbf{R}$-dominating* if $D = \{y_i : i \in S\}$ and, for any $x \in X$, there is some $i_x \in S$ such that $x \sqsubset y_i$ for any $i \geq i_x$ in S.

Remark 2.7. Let $S = \langle S, \leq \rangle$ be a directed partial order. Denote $\mathrm{cp}(S) = \mathfrak{b}(S)$ and $\mathrm{cf}(S) = \mathfrak{d}(S)$, when S is understood as the relational system $\langle S, S, \leq \rangle$. Fix a relational system $\mathbf{R} = \langle X, Y, \sqsubset \rangle$.

(1) The existence of a strongly S-$\mathbf{R}$-dominating family is equivalent to $\mathbf{R} \preceq_{\mathrm{T}} S$. Thus, if such a family exists then $\mathrm{cp}(S) \leq \mathfrak{b}(\mathbf{R})$ and $\mathfrak{d}(\mathbf{R}) \leq \mathrm{cf}(S)$.

(2) If S' is a cofinal subset of S, then any strongly S-$\mathbf{R}$-dominating family contains a strongly S'-$\mathbf{R}$-dominating family.

(3) Any strongly S-$\mathbf{R}$-dominating family is $\mathrm{cp}(S)$-$\mathbf{R}$-dominating of size $\leq |S|$.

(4) Let θ be a cardinal and L a well ordered set with order type θ. If $D \subseteq Y$ has size θ, then D is strongly θ-$\mathbf{R}$-dominating (in the sense of Definition 2.3(5)) iff it is strongly L-$\mathbf{R}$-dominating (in the sense of Definition 2.6).

3. Forcing and elementary submodels

In this section we present the necessary tools to prove the main theorem. From now on, we only deal with the following special type of relational systems.

Definition 3.1. We say that a relational system $\mathbf{R} = \langle X, Y, \sqsubset \rangle$ is *real-definable* if both X and Y are Polish spaces and $\sqsubset$ is Borel in $X \times Y$.[2]

Note that the relational systems in Example 2.2 (in (2) only when $\mathcal{H}$ is countable) are real-definable.

Throughout this section, fix a real-definable relational system $\mathbf{R} = \langle X, Y, \sqsubset \rangle$. From now on, when dealing with $\mathbf{R}$ inside some model N we look at its interpretation $\mathbf{R}^N := \langle X^N, Y^N, \sqsubset^N \rangle$ in the model, but the upper index N is usually omitted due to absoluteness.

The following properties are originally defined in [12, 10, 9].

Definition 3.2. Fix a poset $\mathbb{P}$

(1) For a set S and a cardinal number κ, the property $\mathrm{DOM}_{\mathbf{R}}(\mathbb{P}, \kappa, S)$ states that there is a sequence $\langle \dot{y}_i : i \in S \rangle$ of $\mathbb{P}$-names of members of Y such that, whenever $\gamma < \kappa$ and $\langle \dot{x}_\xi : \xi < \gamma \rangle$ is a sequence of $\mathbb{P}$-names of members of X, there is some $i \in S$ such that $\Vdash \dot{x}_\xi \sqsubset \dot{y}_i$ for all $\xi < \gamma$.

(2) For a directed set S, the property $\mathrm{COB}_{\mathbf{R}}(\mathbb{P}, S)$ states that there is a sequence $\langle \dot{y}_i : i \in S \rangle$ of $\mathbb{P}$-names of members of Y such that, whenever $\dot{x}$ is a $\mathbb{P}$-name of a member of X, there is some $i_* \in S$ such that $\Vdash \dot{x} \sqsubset \dot{y}_i$ for all $i \geq i_*$ in S.

(3) For a linear order L, the property $\mathrm{LCU}_{\mathbf{R}}(\mathbb{P}, L)$ states that there is a sequence $\langle \dot{x}_i : i \in L \rangle$ of $\mathbb{P}$-names of members of X such that, whenever $\dot{y}$ is a $\mathbb{P}$-name of a member of Y, there is some $i_* \in L$ such that $\Vdash \dot{x}_i \not\sqsubset \dot{y}$ for any $i \geq i_*$ in L.

Remark 3.3. It is clear that

[2] In general, we only need that $\sqsubset$ is *sufficiently absolute* in the sense that the statement "$x \sqsubset y$" is absolute between the models we are dealing with.

(1) $\mathrm{LCU}_{\mathbf{R}}(\mathbb{P}, L)$ is equivalent to $\mathrm{COB}_{\mathbf{R}^\perp}(\mathbb{P}, L)$.

(2) $\mathrm{COB}_{\mathbf{R}}(\mathbb{P}, S)$ implies $\mathrm{DOM}_{\mathbf{R}}(\mathbb{P}, \mathrm{cp}(S), S)$.

Remark 3.4. Let S be a directed partial order. If S' is a cofinal subset of S then $\mathrm{COB}_{\mathbf{R}}(\mathbb{P}, S)$ is equivalent to $\mathrm{COB}_{\mathbf{R}}(\mathbb{P}, S')$. Hence, by Remark 3.3(1), if L is a linear order and L' is a cofinal subset of L, $\mathrm{LCU}_{\mathbf{R}}(\mathbb{P}, L)$ is equivalent to $\mathrm{LCU}_{\mathbf{R}}(\mathbb{P}, L')$.

Remark 3.5. In the notation of Definition 3.2,

(a) $\mathrm{DOM}_{\mathbf{R}}(\mathbb{P}, \kappa, S)$ implies that $\mathbb{P}$ forces that $\{\dot{y}_i : i \in S\}$ is a κ-$\mathbf{R}$-dominating family, in particular, $\kappa \leq \mathfrak{b}(\mathbf{R})$ and $\mathfrak{d}(\mathbf{R}) \leq |S|$.

(b) $\mathrm{COB}_{\mathbf{R}}(\mathbb{P}, S)$ implies that $\mathbb{P}$ forces that $\mathbf{R} \preceq_{\mathrm{T}} S$ (i.e., $\{\dot{y}_i : i \in S\}$ is a strongly S-$\mathbf{R}$-dominating family), in particular, $\mathrm{cp}(S) \leq \mathfrak{b}(\mathbf{R})$ and $\mathfrak{d}(\mathbf{R}) \leq \mathrm{cf}(S)$.

(c) $\mathrm{LCU}_{\mathbf{R}}(\mathbb{P}, L)$ implies that $\mathbb{P}$ forces that $\mathbf{R}^\perp \preceq_{\mathrm{T}} L$, in particular, $\mathfrak{b}(\mathbf{R}) \leq \mathrm{cf}(L) \leq \mathfrak{d}(\mathbf{R})$.

However, the converse of these statements only hold in very specific situations.

(1) If S is a set in the ground model and $\mathbb{P}$ forces that there is some κ-$\mathbf{R}$-dominating family $\{y_i : i \in S\}$, then $\mathrm{DOM}_{\mathbf{R}}(\mathbb{P}, \kappa, \ddot{S})$ where $\ddot{S}$ denotes the set of nice $\mathbb{P}$-names of members of S. In the case that $\mathbb{P}$ has ccc, $|\ddot{S}| \leq (|\mathbb{P}| \cdot |S|)^{\aleph_0}$, so $\mathrm{DOM}_{\mathbf{R}}(\mathbb{P}, S)$ holds when $\aleph_0 \leq |\mathbb{P}| \leq |S| = |S|^{\aleph_0}$.

(2) The converse of (b) is true whenever $\mathbb{P}$ has $\mathrm{cp}(S)$-cc, in particular when $\mathbb{P}$ has ccc and $\mathrm{cp}(S)$ is uncountable.

(3) By (2) and Remark 3.3(1), the converse of (c) is true when $\mathbb{P}$ has $\mathrm{cf}(L)$-cc, in particular when $\mathbb{P}$ has ccc and $\mathrm{cf}(L)$ is uncountable.

In our applications the converses of (b) and (c) hold because we only consider ccc posets and directed partial orders S with $\mathrm{cp}(S)$ uncountable.

The following lemmas are the main tools to prove the main theorem.

Lemma 3.6. *Let κ be an uncountable regular cardinal, S a directed partial order, $\mathbb{P}$ a κ-cc poset, and let $N \preceq H_\chi$ (with χ regular large enough) be $<\kappa$-closed with $\mathbb{P}, \kappa, S \in N$. Then:*

(a) $\mathbb{P} \cap N$ is a κ-cc complete subposet of $\mathbb{P}$.

(b) $\mathrm{cp}(S \cap N) \geq \min\{\kappa, \mathrm{cp}(S)\}$.

(c) If $\mathrm{cf}(S) \subseteq N$ then $S \cap N$ is cofinal in S, in particular $\mathrm{cp}(S \cap N) = \mathrm{cp}(S)$ and $\mathrm{cf}(S \cap N) = \mathrm{cf}(S)$.

Proof. Since $\mathbb{P} \cap N$ is a subposet of $\mathbb{P}$ that preserves incompatible conditions, it is clear that $\mathbb{P} \cap N$ is also κ-cc. Now, if A is a maximal antichain in $\mathbb{P} \cap N$, then $A \in N$ because $|A| < \kappa$ and N is $<\kappa$-closed. Hence $N \models$"A is a maximal antichain in $\mathbb{P}$", so the same holds in the universe. Therefore, $\mathbb{P} \cap N \lessdot \mathbb{P}$. This shows (a).

To see (b), if $F \subseteq S \cap N$ has size $< \min\{\kappa, \mathrm{cp}(S)\}$, then $F \in N$, so $N \models$"$(\exists b \in S)(\forall a \in F)\, a \leq b$", hence there is some $b \in S \cap N$ that bounds $F \cap N = F$ from above.

For (c), since $\mathrm{cf}(S) \in N$, we can find some $S' \in N$ cofinal in S of size $\mathrm{cf}(S)$. But also $\mathrm{cf}(S) \subseteq N$, so $S' \subseteq N$. Hence $S \cap N \supseteq S'$, so $S \cap N$ is cofinal in S. □

Lemma 3.7. *With the same hypothesis as in the previous lemma, if in addition the parameters defining* $\mathbf{R}$ *are in* N*,* $\nu \in N$ *is a cardinal and* $D \in N$*, then:*

(a) $\mathrm{DOM}_{\mathbf{R}}(\mathbb{P}, \nu, D)$ *implies* $\mathrm{DOM}_{\mathbf{R}}(\mathbb{P} \cap N, \min\{\kappa, \nu\}, D \cap N)$*.*

(b) $\mathrm{COB}_{\mathbf{R}}(\mathbb{P}, S)$ *implies* $\mathrm{COB}_{\mathbf{R}}(\mathbb{P} \cap N, S \cap N)$*.*

(c) If $|D| \subseteq N$ *then* $\mathrm{DOM}_{\mathbf{R}}(\mathbb{P}, \nu, D)$ *implies* $\mathrm{DOM}_{\mathbf{R}}(\mathbb{P} \cap N, \nu, D)$*.*

(d) If $\mathrm{cf}(S) \subseteq N$ *then* $\mathrm{COB}_{\mathbf{R}}(\mathbb{P}, S)$ *implies* $\mathrm{COB}_{\mathbf{R}}(\mathbb{P} \cap N, S)$*.*

Proof. Assume $\mathrm{DOM}_{\mathbf{R}}(\mathbb{P}, \nu, D)$. Since this property also holds in N, we can find $\langle \dot{y}_i : i \in D\rangle \in N$ witnessing DOM. To show (a), we prove that $\langle \dot{y}_i : i \in D \cap N\rangle$ witnesses $\mathrm{DOM}_{\mathbf{R}}(\mathbb{P} \cap N, \min\{\kappa, \nu\}, D \cap N)$. Since any member of a Polish space can be coded by a real, and nice names of reals are coded by countably many maximal antichains, if wlog we assume that each $\dot{y}_i$ is a nice $\mathbb{P}$-name, then the maximal antichains coding it are in N because N is $<\kappa$-closed, so $\dot{y}_i \in N$ and it is in fact a $\mathbb{P} \cap N$-name.

Let $\gamma < \min\{\kappa, \nu\}$ and $\bar{x} := \langle \dot{x}_\alpha : \alpha < \gamma\rangle$ be a sequence of nice $\mathbb{P} \cap N$-names of members of X. Then, they are also $\mathbb{P}$-names and, since N is $<\kappa$-closed, $\bar{x} \in N$. Hence $N \models$"$(\exists i \in D)(\forall \alpha < \gamma) \Vdash_{\mathbb{P}} \dot{x}_\alpha \sqsubset \dot{y}_i$", so there is some $i \in D \cap N$ such that $\Vdash_{\mathbb{P}} \dot{x}_\alpha \sqsubset \dot{y}_i$ for any $\alpha < \gamma$. Since $\mathbb{P} \cap N$ is a complete subposet of $\mathbb{P}$, we are done. A similar argument shows (b).

To see (c), note that $D \in N$ and $|D| \subseteq N$ implies $D \subseteq N$, hence any witness of $\mathrm{DOM}_{\mathbf{R}}(\mathbb{P}, \nu, D)$ (formed by nice $\mathbb{P}$-names), is a witness of $\mathrm{DOM}_{\mathbf{R}}(\mathbb{P} \cap N, \nu, D)$. Property (d) can be verified likewise. □

Lemma 3.8. *Let* $\kappa \leq \lambda \leq \theta$ *be cardinals with* κ *and* λ *uncountable regular,* S *a directed set and* $\mathbb{P}$ *a* κ*-cc poset. Assume that* $\langle N_\xi : \xi < \lambda\rangle$ *is an increasing sequence of* $<\kappa$*-closed elementary submodels of* H_χ *of size* θ*, containing* $\theta \cup \{\theta, \mathbb{P}, S\}$ *and the parameters of* $\mathbf{R}$*, such that* $N_\xi \in N_{\xi+1}$ *for any* $\xi < \lambda$*. Set* $N := \bigcup_{\xi<\lambda} N_\xi$ *(which is also a* $<\kappa$*-closed elementary submodel of* H_χ*). Then:*

(a) If $\mathrm{cp}(S) > \theta$ *then* $\mathrm{cf}(S \cap N) = \lambda$*.*

(b) If $\mu \in N$ *is regular then* $\mathrm{cf}(\mu \cap N) = \mu$ *whenever* $\mu \leq \theta$*, otherwise* $\mathrm{cf}(\mu \cap N) = \lambda$*.*

(c) If $\nu \in N$ *is a cardinal,* $\nu > \theta$ *and* $D \in N$*, then* $\mathrm{DOM}_{\mathbf{R}}(\mathbb{P}, \nu, D)$ *implies* $\mathrm{COB}_{\mathbf{R}}(\mathbb{P} \cap N, \lambda)$*.*

(d) If $\mathrm{cp}(S) > \theta$ *then* $\mathrm{COB}_{\mathbf{R}}(\mathbb{P}, S)$ *implies* $\mathrm{COB}_{\mathbf{R}}(\mathbb{P} \cap N, \lambda)$*.*

Proof. We show (a). For each $\xi < \lambda$, $|S \cap N_\xi| \leq \theta < \mathrm{cp}(S)$, so there is some $i_\xi \in S$ that bounds $S \cap N_\xi$ from above. In fact, we can find such $i_\xi \in S \cap N_{\xi+1}$ because $N_\xi, S \in N_{\xi+1}$. Hence $\langle i_\xi : \xi < \lambda\rangle$ is a cofinal increasing sequence in $S \cap N$, so $\mathrm{cf}(S \cap N) = \mathrm{cp}(S \cap N) = \lambda$.

For (b), when $\mu \leq \theta$ we have $\mu \subseteq N$, so $\mathrm{cf}(\mu \cap N) = \mathrm{cf}(\mu) = \mu$; otherwise, applying (a) to $S := \mu$ we get $\mathrm{cf}(\mu \cap N) = \lambda$.

For (c), let $\ddot{X}$ be the set of nice $\mathbb{P}$-names of members of X. For each $\xi < \lambda$, $|\ddot{X} \cap N_\xi| \leq \theta < \nu$, so $\mathrm{DOM}_{\mathbf{R}}(\mathbb{P}, \nu, D)$, witnessed by $\langle \dot{y}_i : i \in D\rangle$ (assuming that they are nice names) implies that there is some $i_\xi \in D$ such that $\Vdash_{\mathbb{P}} \dot{x} \sqsubset \dot{y}_{i_\xi}$ for any $\dot{x} \in \ddot{X} \cap N_\xi$, in fact, we can find $i_\xi \in N_{\xi+1}$. Hence, $\langle \dot{y}_{i_\xi} : \xi < \lambda\rangle$ witnesses $\mathrm{COB}_{\mathbf{R}}(\mathbb{P} \cap N, \lambda)$.

Property (d) follows from (c) because $\mathrm{COB}_{\mathbf{R}}(\mathbb{P}, S)$ implies $\mathrm{DOM}_{\mathbf{R}}(\mathbb{P}, \mathrm{cp}(S), S)$ by Remark 3.3(2). □

4. Proof of the main theorem

We denote the relational systems introduced in Example 2.2 by $\mathbf{R}_1 := \mathbf{Lc}$, $\mathbf{R}_2 := \mathbf{Cn}$, $\mathbf{R}_3 := \mathbf{D}$ and $\mathbf{R}_4 := \mathbf{Mg}$. For $i = 1, 2, 3, 4$, COB_i abbreviates $\mathrm{COB}_{\mathbf{R}_i}$, likewise for DOM_i and LCU_i.

The starting point to prove the main theorem is to find a ccc poset forcing, through the properties LCU and COB (even DOM is just fine), that the cardinals on the left side of Cichoń's diagram are pairwise different. This was done in [7, 10, 4]. For the author's convenience (and also to deal with a short hypothesis), we use the construction in [4].

Theorem 4.1 ([4, Thm. 5.3]). *Let $\langle \nu_j : j = 1, \dots, 6\rangle$ be a monotone increasing sequence of uncountable cardinals such that ν_j is regular for $j = 1, \dots, 5$ and $\nu_6^{<\nu_3} = \nu_6$. Then there is a ccc poset $\mathbb{P}^-$ of size ν_6 such that:*

(1) $\mathrm{LCU}_i(\mathbb{P}^-, \kappa)$ holds for any regular $\theta_i \leq \kappa \leq \theta_6$ and $i = 1, 2, 3$; for $i = 4$, $\mathrm{LCU}_4(\mathbb{P}^-, \nu_4)$ and $\mathrm{LCU}_4(\mathbb{P}^-, \nu_5)$ hold.

(2) For each $i = 1, 2, 3$ there is a directed order S_i with $\mathrm{cp}(S_i) = \nu_i$ and $\mathrm{cf}(S_i) = |S_i| = \nu_6$ such that $\mathrm{COB}_i(\mathbb{P}^-, S_i)$ holds; for $i = 4$, there is a directed order S_4 with $\mathrm{cp}(S_4) = \nu_4$ and $\mathrm{cf}(S_4) = |S_4| = \nu_5$ such that $\mathrm{COB}_4(\mathbb{P}^-, S_4)$ holds.

In particular, $\mathbb{P}^-$ forces $\mathrm{add}(\mathcal{N}) = \nu_1 \leq \mathrm{cov}(\mathcal{N}) = \nu_2 \leq \mathfrak{b} = \nu_3 \leq \mathrm{non}(\mathcal{M}) = \nu_4 \leq \mathrm{cov}(\mathcal{M}) = \nu_5 \leq \mathfrak{d} = \mathrm{non}(\mathcal{N}) = \mathfrak{c} = \nu_6$.

Now we are ready to prove the main theorem. Note that it would be enough to assume GCH, but for our convenience the result is presented with weaker hypothesis (which could be even weaker, see [9, Rem. 3.3]).

Theorem 4.2 ([9, Thm. 3.1]). *Let $\langle \mu_j : j = 1, \dots, 9\rangle$ be a monotone increasing sequence of uncountable cardinals such that μ_j is regular for $1 \leq j \leq 8$ and $\mu_9^{\aleph_0} = \mu_9$. In addition, assume that there are at least 9 cardinals $\theta > \mu_9$ such that $\theta^{<\theta} = \theta$. Then there is a ccc poset that forces $\mathrm{add}(\mathcal{N}) = \mu_1 \leq \mathrm{cov}(\mathcal{N}) = \mu_2 \leq \mathfrak{b} = \mu_3 \leq \mathrm{non}(\mathcal{M}) = \mu_4 \leq \mathrm{cov}(\mathcal{M}) = \mu_5 \leq \mathfrak{d} = \mu_6 \leq \mathrm{non}(\mathcal{N}) = \mu_7 \leq \mathrm{cof}(\mathcal{N}) = \mu_8 \leq \mathfrak{c} = \mu_9$.*

Proof. Fix a sequence of cardinals

$$\aleph_1 \leq \lambda_7 \leq \lambda_5 \leq \lambda_3 \leq \lambda_1 \leq \lambda_0 \leq \lambda_2 \leq \lambda_4 \leq \lambda_6 \leq \lambda_\infty \\ < \theta_7 < \theta_6 < \theta_5 < \theta_4 < \theta_3 < \theta_2 < \theta_1 < \theta_0 < \theta_\infty$$

such that

(1) $\lambda_{8-2i+1} := \mu_i$ and $\lambda_{8-2i} := \mu_{9-i}$ for $i = 1, 2, 3, 4$, and $\lambda_\infty := \mu_9$;

(2) $\theta_n^{<\theta_n} = \theta_n$ for all $n = \infty, 0, \dots, 7$.

Put $\nu_i := \theta_{8-2j}$ for each $i = 1, 2, 3, 4$ and $\nu_5 = \nu_6 := \theta_\infty$, and let $\mathbb{P}^-$ be the ccc poset resulting from the application of Theorem 4.1 to these values.

Choose a regular cardinal χ large enough, and for $0 \leq n \leq 7$ and $\alpha < \lambda_n$, fix $N_{n,\alpha}$ and N_8 satisfying the following (this can be done by (2) and because $\lambda_\infty^{\aleph_0} = \lambda_\infty$):

(i) $N_{n,\alpha}$ and N_8 are elementary submodels of H_χ containing (as elements) $\mathbb{P}^-$, the sequences of θ's and λ's, as well as the directed orders S_i obtained from Theorem 4.1.

(ii) $\langle N_{m,\beta} : m < n,\ \beta < \lambda_m \rangle, \langle N_{n,\beta} : \beta < \alpha \rangle \in N_{n,\alpha}$ and $\langle N_{m,\beta} : m \leq 7,\ \beta < \lambda_m \rangle \in N_8$.

(iii) $N_{n,\alpha}$ is $<\theta_n$-closed of size θ_n, and $\theta_n \subseteq N_{n,\alpha}$.

(iv) N_8 is $<\aleph_1$-closed of size λ_∞, and $\lambda_\infty \subseteq N_8$.

Set $N_n := \bigcup_{\alpha<\lambda_n} N_{n,\alpha}$, which is a $<\lambda_n$-closed elementary submodel of H_χ of size θ_n. For $0 \leq m \leq 8$, define $N^*_m := \bigcap_{n=0}^{m} N_m$ and $\mathbb{P}_m := \mathbb{P} \cap N^*_m$. Note that $N^*_m \preceq H_\chi$,[3] moreover, N^*_m is $<\aleph_1$-closed, so $\mathbb{P}_n \lessdot \mathbb{P}_m \lessdot \mathbb{P}^-$ for $m < n$ by Lemma 3.6(a).

We show that $\mathbb{P}_8$ is the required poset, in fact, we show that $\mathrm{LCU}_i(\mathbb{P}_8, \lambda_{8-2i})$, $\mathrm{LCU}_i(\mathbb{P}_8, \lambda_{8-2i+1})$ and $\mathrm{COB}_i(\mathbb{P}_8, S_i \cap N^*_8)$ hold for any $i = 1, 2, 3, 4$, with $\lambda_{8-2i+1} \leq \mathrm{cp}(S_i \cap N^*_8)$ and $\mathrm{cf}(S_i \cap N^*_8) \leq \lambda_{8-2i}$ (so the result follows by Remark 3.5(b),(c)).

Fix $1 \leq i \leq 4$. By Theorem 4.1, $\mathrm{LCU}_i(\mathbb{P}^-, \theta_\infty)$ and $\mathrm{LCU}_i(\mathbb{P}^-, \theta_j)$ hold for $j \leq 8-2i$. By induction on $m \leq 8$, we show that $\mathrm{LCU}_i(\mathbb{P}_m, \lambda_j)$ holds for $j \leq \min\{m, 8-2i+1\}$, and $\mathrm{LCU}_i(\mathbb{P}_m, \theta_j)$ holds for $\min\{m, 8-2i+1\} \leq j \leq 8-2i$ (this second statement is vacuous when $m \geq 8-2i+1$). The case $m = 8$ gives us the desired result about LCU.

Case $m = 0$. Note that N^*_0 is $<\lambda_0$-closed of size θ_0. Since $\theta_0 \cup \{\theta_0\} \subseteq N^*_0$ (by (i) and (iii)), $\mathrm{LCU}_i(\mathbb{P}_0, \theta_j)$ holds for any $j \leq 8-2i$ by Lemma 3.7(d) and Remark 3.3(1); on the other hand, by Lemma 3.8(d), $\mathrm{LCU}_i(\mathbb{P}^-, \theta_\infty)$ implies $\mathrm{LCU}_i(\mathbb{P}_0, \lambda_0)$.

Successor step. Assume our claim holds for $m < 8$. Note that $N^*_m \in N^*_{m+1}$, so $\mathbb{P}_m \in N^*_{m+1}$. First assume that $m \leq 8-2i$, so $\mathrm{LCU}_i(\mathbb{P}_m, \lambda_j)$ holds for $j \leq m$ and $\mathrm{LCU}_i(\mathbb{P}_m, \theta_j)$ holds for $m \leq j \leq 8-2i$. Since $\lambda_j < \theta_{m+1}$ for $j \leq m$, and $\theta_{j'} \leq \theta_{m+1}$ for $m+1 \leq j' \leq 8-2i$, $\mathrm{LCU}_i(\mathbb{P}_{m+1}, \lambda_j)$ and $\mathrm{LCU}_i(\mathbb{P}_{m+1}, \theta_{j'})$ hold by Lemma 3.7(d); on the other hand, $\mathrm{LCU}_i(\mathbb{P}_m, \theta_m)$ implies $\mathrm{LCU}_i(\mathbb{P}_{m+1}, \lambda_{m+1})$ by Lemma 3.8(d).

Now assume that $m \geq 8-2i+1$, so $\mathrm{LCU}_i(\mathbb{P}_m, \lambda_j)$ holds for $j \leq 8-2i+1$. Since for all such j, $\lambda_j \leq \theta_{m+1}$ when $m < 7$, and $\lambda_j \leq \lambda_\infty$ in case $m = 7$, by Lemma 3.7(d) we have $\mathrm{LCU}_i(\mathbb{P}_{m+1}, \lambda_j)$.

Regarding COB, fix $1 \leq i \leq 4$ and put $m := 8-2i$. Recall that $\mathrm{COB}_i(\mathbb{P}^-, S_i)$ holds by Theorem 4.1. We first show that $\mathrm{cp}(S_i \cap N^*_{m+1}) \geq \lambda_{m+1}$ and $\mathrm{cf}(S_i \cap N^*_{m+1}) \leq \lambda_m$. Using Lemma 3.6(b) it is easy to show by induction on $n \leq m+1$ that $\mathrm{cp}(S_i \cap N^*_n) \geq \lambda_n$. Concerning the cofinality, fix $\Lambda := \prod_{n=0}^{m} \lambda_n$ and, for each $\eta \in \Lambda$, set $N^\eta := \bigcap_{n=0}^{m} N_{n,\eta(n)}$. Note that $N^*_m = \bigcup_{\eta\in\Lambda} N^\eta$, and that Λ is element and subset of any of the considered submodels of H_χ.

For each $\eta \in \Lambda$, since N^η is $<\theta_m$-closed and $\mathrm{cp}(S_i) \geq \theta_m$, we get $\mathrm{cp}(S_i \cap N^\eta) \geq \theta_m > \theta_{m+1}$. Hence, by Lemma 3.8(a), $\mathrm{cf}(S_i \cap N^\eta \cap N_{m+1}) = \lambda_{m+1}$. Choose $C_\eta \subseteq S_i \cap N^\eta \cap N_{m+1}$ cofinal of size λ_{m+1}, so $C := \bigcup_{\eta\in\Lambda} C_\eta$ is cofinal in $S_i \cap N^*_{m+1} = \bigcup_{\eta\in\Lambda} S_i \cap N^\eta \cap N_{m+1}$. Therefore, $\mathrm{cf}(S_i \cap N^*_{m+1}) \leq |C| \leq \lambda_m$.

Now, by induction on n with $m+1 \leq n \leq 8$, we show that $\mathrm{cp}(S_i \cap N^*_n) \geq \lambda_{m+1}$ and $\mathrm{cf}(S_i \cap N^*_n) \leq \lambda_m$. The case $n = m+1$ was already taken care of. For the inductive step, since $\mathrm{cf}(S_i \cap N^*_n) \leq \lambda_m$ we have that $S_i \cap N^*_{n+1}$ is cofinal in $S_i \cap N^*_n$ by Lemma 3.6(c), in particular $\mathrm{cp}(S_i \cap N^*_{n+1}) = \mathrm{cp}(S_i \cap N^*_n) \geq \lambda_{m+1}$ and $\mathrm{cf}(S_i \cap N^*_{n+1}) = \mathrm{cf}(S_i \cap N^*_n) \leq \lambda_m$. When $n = 8$ we obtain $\mathrm{COB}_i(\mathbb{P}_8, S_i \cap N^*_8)$ by Lemma 3.7(b).

To finish, we show that $\mathbb{P}_8$ forces $\mathfrak{c} = \lambda_\infty$. It is clear that $|\mathbb{P}_8| \leq \lambda_\infty$, so $\mathbb{P}_8$ forces $\mathfrak{c} \leq |\mathbb{P}_8|^{\aleph_0} \leq \lambda_\infty$. For the converse inequality, since $\mathbb{P}^-$ forces $\mathfrak{c} = \theta_\infty$ it is clear that $\mathrm{LCU}_{\mathbf{Id}}(\mathbb{P}^-, \kappa)$ holds for any infinite $\kappa \leq \theta_\infty$, in particular, for any regular $\kappa \leq \lambda_\infty$. By Lemma 3.7(d), $\mathrm{LCU}_{\mathbf{Id}}(\mathbb{P}_8, \kappa)$ holds for any regular $\kappa \leq \lambda_\infty$, so $\mathbb{P}_8$ forces $\mathfrak{c} \geq \lambda_\infty$. □

[3] If $M, N \preceq H_\chi$ and $M \in N$ then $M \cap N \preceq M$ and $M \cap N \preceq N$.

Remark 4.3. The construction of the forcing of Theorem 4.1 is slightly simpler when, instead of (2), it is demanded:

(2^-) For each $i = 1, 2, 3$, $\mathrm{DOM}_i(\mathbb{P}^-, \nu_i, \nu_6)$ holds; for $i = 4$, $\mathrm{COB}_4(\mathbb{P}^-, \nu_4, \nu_5)$ holds.

The proof of Theorem 4.2 can be carried out in this simpler context.

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