特別講演

Hyperbolic solutions to Bernoulli's free boundary problem

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1. The Bernoulli problem

Let Ω be a bounded domain in $\mathbb{R}^2$ and $Q > 0$ a constant. The Bernoulli problem is the problem of finding an open subset $A \subset \Omega$ for which the following overdetermined problem is solvable:

$$\begin{cases} -\Delta u = 0 & \text{in} \quad \Omega \setminus \overline{A}, \\ u = 0 & \text{on} \quad \partial\Omega, \\ u = 1 & \text{on} \quad \partial A, \\ \dfrac{\partial u}{\partial \nu} = Q & \text{on} \quad \partial A, \end{cases} \tag{1.1}$$

where ν is the unit outer normal vector with respect to the annular domain $\Omega \setminus \overline{A}$. The first three equations comprise the classical Dirichlet problem, and it has a unique solution u. Thus (1.1) has an extra boundary condition, which makes a restriction on A for the solvability of (1.1).

Equations (1.1) arises in a shape optimization problem in which one wants to design the shape of the insulation layer of an electronic cable such that the current leakage is minimized subject to a given amount of insulation material. Then, u stands for the electrostatic potential and Ω is the cross-section of the cable with the insulation layer $\Omega \setminus \overline{A}$. Another physical interpretation of u is the stream potential of the stationary irrotational velocity field in the plane of an incompressible inviscid fluid which circulates around a bubble A of air in a given container Ω.

The existence of a solution A for prescribed Ω and Q is shown by various methods including the super and subsolution method of Beurling [3], a variational method by Alt and Caffarelli [2], and the inverse function theorem of Nash and Moser by Hamilton [6]. However, most of the results are concerned with a class of "stable", or "well-ordered", solutions called elliptic solutions (see Definition 2), where a solution A to the Bernoulli problem is called elliptic if, roughly speaking, the infinitesimal increase of the value of $Q > 0$ makes the corresponding solution A to expand. Indeed, the super and subsolution method only allows one to construct elliptic solutions, since the method constructs a solution A as the union of all subsolutions, where $A_{\mathrm{sub}} \subset \Omega$ is called a subsolution to (1.1) if there exists a solution u to (1.1) in $\Omega \setminus A_{\mathrm{sub}}$ with the last boundary condition replaced by $\partial_\nu u \leq Q$; and hence for $\tilde{Q} > Q$ the corresponding solution $\tilde{A}$ must be larger than A. Variational solutions constructed as minimizers are also elliptic. This can be seen by looking at the form of the variational functional (see [5, Section 5.3]). On the other hand, the inverse function theorem is, in principle, able to handle "unstable" solutions called hyperbolic solutions, for which the increase of $Q > 0$ makes A to shrink; but (1.1) has a regularity issue called "loss of derivatives", and this requires several estimates which are only (at least up to now) available for elliptic solutions.

The structure of solutions A to the Bernoulli problem is illustrated by the simplest situation where Ω is the unit ball $\mathbb{B} = \mathbb{B}_1$, Here, we denote by $\mathbb{B}_r$ the ball of radius $r > 0$ with center at the origin.

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Example . For $\Omega = \mathbb{B}$, it is known that solutions A must be concentric balls. The function u_r satisfying the first three equations in (1.1) for $A = \mathbb{B}_r$ $(0 < r < 1)$ and its normal derivative at $|x| = r$ are

$$u_r(x) = \frac{\log|x|}{\log r}, \quad \frac{\partial u_r}{\partial \nu}(r) = -\frac{1}{r\log r}.$$

The function $Q(r) := \partial_\nu u_r(r)$ $(0 < r < 1)$ is convex and takes its minimum at $r = 1/e$. Therefore, for $Q = Q_0 := Q(1/e)$, the Bernoulli problem has a unique solution $A = \mathbb{B}_{1/e}$; while there are an elliptic solution $\mathbb{B}_{r_e(Q)}$ and a hyperbolic solution $\mathbb{B}_{r_h(Q)}$ with $r_e(Q) > 1/e > r_h(Q)$ for $Q > Q_0$; and no solution for $Q < Q_0$. Moreover, these solutions satisfy

$$\lim_{Q\to\infty} r_e(Q) = 1, \quad \lim_{Q\to\infty} r_h(Q) = 0.$$

Thus, the elliptic solutions $\mathbb{B}_{r_e(Q)}$ asymptotically approach to the prescribed domain $\Omega = \mathbb{B}$, and the hyperbolic solutions $\mathbb{B}_{r_h(Q)}$ shrink to the single point $\{0\}$.

One of the interesting questions is whether such a foliation structure of solutions can be observed for general convex domain Ω. As a matter of fact, Acker [1] proved that this is true for elliptic solutions. Our research was initiated towards the affirmative answer to the following conjecture.

Conjecture (Flucher and Rumpf [5])**.** *There exist hyperbolic foliated solutions shrinking to the conformal center $z_0 \in \Omega$ for any convex domain Ω.*

The conformal centers of a simply-connected domain Ω are the points $z_0 \in \Omega$ at which the conformal radius

$$R(z) := |f_z'(0)| \quad (z \in \Omega)$$

takes its maximum, where $f_z : \mathbb{B} \to \Omega$ is the biholomorphic map satisfying $f(0) = z$ and $f'(0) > 0$. It is known that the conformal radius $R(z)$ is strictly concave if Ω is convex; and thus the conformal center is unique (see Cardaliaguet and Tahraoui [4]).

This conjecture has been open for decades. This is due to the fact that many arguments based on the maximum principles or the variational method fail for hyperbolic solutions. The inverse function theorem can possibly apply to capture hyperbolic solutions; however, the problem of "loss of derivatives" forced one to work with not only a Banach space, but also a graded family of spaces (see [6]), and solutions in general have lower regularity than the initial solution does. Thus, these existing methods are not suitable to pursue the behavior of hyperbolic solutions A when the value $Q > 0$ changes.

The main contribution of this work is to introduce a new "parabolic" approach, namely that we derive and analyze a flow equation describing the behavior of solutions $A = A(t)$ for varying data $Q = Q(t)$. As a result, we are able to construct locally foliated hyperbolic solutions. This approach has a common feature with the inverse function theorem in [6], since the derivation of the flow equation is essentially based on the linearization of (1.1). But the parabolic approach has the advantage that "loss of derivatives" can be handled with the established theory of evolution equations; and hence a graded family of spaces is no longer needed and we can work with a fixed Banach space. Moreover, our method can apply in any space dimensions $n \geq 2$, and thus hereafter we consider (1.1) in $\mathbb{R}^n$.

2. Deformation flow

To explain how our approach can handle the regularity problem, let us consider the abstract functional equation

$$F(x,s)=0 \quad (x\in X,\ s\in\mathbb{R}), \tag{2.1}$$

where X is a Banach space and F is a C^1-mapping from $X\times\mathbb{R}$ to another Banach space Y with $X\subset Y$. If $F(0,0)=0$ and the Fréchet derivative $\partial_x F(0,0)\in\mathcal{L}(X,Y)$ (the space of bounded operators from X to Y) is invertible, then for each given small data s we can find a unique solution $x(s)\in X$ in a neighborhood of 0 by the implicit function theorem. In fact, the sequence of X-valued curves

$$x_1(s):=0,\quad x_{j+1}(s):=x_j(s)-\partial_x F(0,0)^{-1}F(x_j(s),s)\quad(-\varepsilon\le s\le\varepsilon) \tag{2.2}$$

converges to a C^1-curve $x(s)$ satisfying $x(0)=0$ and $F(x(s),s)=0$. However, the method would fail if we only have the regularity gain $\partial_x F(0,0)^{-1}\in\mathcal{L}(Y,Y)^\dagger$; and thus $x_{j+1}(s)$ is merely Y-valued even if $x_j(s)$ is X-valued. This "loss of derivatives" actually happens in the Bernoulli problem (1.1), and one is required to use the technique of Nash and Moser to overcome this regularity issue. We, instead, consider the evolution equation

$$x'(s)=-\partial_x F(x(s),s)^{-1}\partial_s F(x(s),s),\quad x(0)=0. \tag{2.3}$$

This equation is formally derived by differentiating (2.1) in s with $x=x(s)$, and it is easy to see that $F(x(s),s)=0$ if and only if $x=x(s)$ is a solution to (2.3). A natural regularity condition for this parabolic formulation is

$$x(s)\in C([0,\varepsilon),X)\cap C^1([0,\varepsilon),Y),$$

so that the "loss of derivatives" is no longer an issue and it is well-treated within the standard theory of evolution equations.

Now we set up the Bernoulli problem (1.1) as a functional equation of the form $F(\rho,Q)=0$. Hereafter, Ω denotes a fixed bounded domain in $\mathbb{R}^n$ with $h^{2+\alpha}$-boundary, and Q is allowed to be a positive function in $h^{2+\alpha}(\mathbb{R}^n)$, where $h^{k+\alpha}(\Gamma)$ is the so-called little Hölder space on a domain (or hypersurface) Γ defined as the closure of $C^\infty(\Gamma)$ in the topology of the Hölder space $C^{k+\alpha}(\Gamma)$. Let us choose a reference domain $A_0\subset\overline{A_0}\subset\Omega$ with smooth boundary ∂A_0, say of class $h^{4+\alpha}$, and identify $\rho\in\mathcal{U}_\gamma\subset h^{3+\alpha}(\partial A_0)$ with a perturbed domain A_ρ having $h^{3+\alpha}$-boundary

$$\partial A_\rho=\{\zeta+\rho(\zeta)\nu(\zeta)\mid\zeta\in\partial A_0\}, \tag{2.4}$$

where $\nu=\nu(\zeta)$ is the unit outer normal vector with respect to $\Omega\setminus\overline{A_0}$ and

$$\mathcal{U}_\gamma:=\{\rho\in h^{3+\alpha}(\partial A_0)\mid\|\rho\|_{h^{3+\alpha}(\partial A_0)}<\gamma\},\quad \gamma\le a/4,$$

with $0<a<\operatorname{dist}(\partial A_0,\partial\Omega)$ taken to be small such that $\theta(\zeta,r):=\zeta+r\nu(\zeta)$ defines a diffeomorphism from $\partial A_0\times(-a,a)$ to its image. Denoting by ζ and r the components of the inverse map θ^{-1}, i.e., $\theta^{-1}(x)=(\zeta(x),r(x))$, we see that

$$\theta_\rho(x):=\begin{cases}\theta\left(\zeta(x),r(x)+\eta(r(x))\rho(\zeta(x))\right) & \text{if } x\in\theta(\partial A_0\times(-a,a)),\\ x & \text{otherwise,}\end{cases}$$

†Of course, this happens only when $\partial_x F(0,0)$ is not invertible. We will give the precise meaning of the "inverse" $\partial_x F(0,0)^{-1}$ later.

defines an $h^{2+\alpha}$-diffeomorphism from $\Omega \setminus A_0$ to the annular domain $\Omega \setminus A_\rho$, where η is a smooth cut-off function satisfying

$$\eta(r) = \begin{cases} 1 & (|r| \le a/4), \\ 0 & (|r| \ge 3a/4) \end{cases} \qquad \text{and} \qquad \left| \frac{d\eta}{dr}(r) \right| < \frac{4}{a}. \tag{2.5}$$

The diffeomorphism θ_ρ induces the pull-back and push-forward operators

$$\theta_\rho^* u := u \circ \theta_\rho, \quad \theta_*^\rho v := v \circ \theta_\rho^{-1}$$

for $u \in h^{k+\alpha}(\overline{\Omega} \setminus A_\rho)$ and $v \in h^{k+\alpha}(\overline{\Omega} \setminus A_0)$ $(0 \le k \le 2)$. For a given $\rho \in \mathcal{U}_\gamma$, the first three equations in (1.1) with $A = A_\rho$ comprise the Dirichlet problem and thus always have a unique solution $u_\rho \in h^{2+\alpha}(\overline{\Omega} \setminus A_\rho)$. Hence if we define

$$F(\rho, Q) := \theta_\rho^* \left(\frac{\partial u_\rho}{\partial \nu} - Q \right),$$

then A_ρ is a solution to (1.1) if and only if $F(\rho, Q) = 0$.

Proposition 1. *Suppose that $Q \in h^{2+\alpha}(\mathbb{R}^n)$ and $\rho \in \mathcal{U}_\gamma$. Then,*

(i) $F \in C^1(\mathcal{U}_\gamma \times \mathbb{R}, h^{1+\alpha}(\partial A_0))$.

(ii) The Fréchet derivative of F with respect to ρ at $\rho = 0$ is given by

$$\partial_\rho F(0, Q)[\tilde{\rho}] = \frac{\partial p}{\partial \nu} - HQ\tilde{\rho} - \frac{\partial Q}{\partial \nu}\tilde{\rho} \quad \left(\tilde{\rho} \in h^{3+\alpha}(\partial A_0)\right),$$

where $H = H_{\partial A_0} \in h^{1+\alpha}(\partial A_0)$ is the mean curvature of ∂A_0 normalized in such a way that $H = -(n-1)$ if $A_0 = \mathbb{B}$, and p is the solution to

$$\begin{cases} -\Delta p = 0 & \text{in} \quad \Omega \setminus \overline{A_0}, \\ p = 0 & \text{on} \quad \partial\Omega, \\ p = -Q\tilde{\rho} & \text{on} \quad \partial A_0. \end{cases} \tag{2.6}$$

(iii) The linear operator given above is extended to

$$\partial_\rho F(0, Q) \in \mathcal{L}\left(h^{2+\alpha}(\partial A_0), h^{1+\alpha}(\partial A_0)\right).$$

The extension $\partial_\rho F(0, Q)$ in Proposition 1 (iii) with $Q(x) > 0$ has the bounded inverse $\partial_\rho F(0, Q)^{-1} \in \mathcal{L}(h^{1+\alpha}(\partial A_0), h^{2+\alpha}(\partial A_0))$ if the elliptic equation

$$\begin{cases} -\Delta p = 0 \quad \text{in} \quad \Omega \setminus \overline{A}, \\ p = 0 \quad \text{on} \quad \partial\Omega, \\ \dfrac{\partial p}{\partial \nu} + \left(H + \dfrac{\partial_\nu Q}{Q} \right) p = q \quad \text{on} \quad \partial A \end{cases} \tag{2.7}$$

with $A = A_0$ is uniquely solvable for any $q \in h^{1+\alpha}(\partial A_0)$. Moreover,

$$\partial_\rho F(0, Q)^{-1}[q] = -\frac{p}{Q} \in h^{2+\alpha}(\partial A_0) \quad \left(q \in h^{1+\alpha}(\partial A_0)\right),$$

where p is the unique solution to (2.7). Let us now recall some notions for solutions A to (1.1) in terms of the linearized problem (2.7).

Definition 1 (Non-degeneracy)**.** *We say that a domain A is non-degenerate if the linearized problem* (2.7) *with $q = 0$ has only the trivial solution $p = 0$.*

Remark 1. The non-degeneracy of A, in fact, guarantees the unique solvability of (2.7) for any inhomogeneous data q by the Fredholm theory.

Furthermore, a classification of solutions A in terms of the behavior of solutions p to (2.7) was introduced by Flucher and Rumpf [5] as an extension of Beurling's original definition [3].

Definition 2 (Elliptic, hyperbolic and parabolic solutions)**.** *A solution A to the Bernoulli problem* (1.1) *is called elliptic (hyperbolic) if* (2.7) *has a solution for $q = 1$ and all the solutions p satisfy*

$$\int_{\partial A} p \, d\sigma > 0 \quad (< 0).$$

Otherwise, A is called parabolic. Moreover, an elliptic (hyperbolic) solution A is said to be monotone if $p > 0$ (< 0) holds everywhere on ∂A.

Remark 2. Elliptic (hyperbolic) solutions are interpreted as volume-increasing (decreasing) solutions $A(\varepsilon)$ for varying $Q(\varepsilon) = Q + \varepsilon$, since

$$\frac{d}{d\varepsilon}\left[\int_{A(\varepsilon)} dx\right] = \frac{1}{Q(\varepsilon)} \int_{\partial A(\varepsilon)} p \, d\sigma > 0 \quad (< 0).$$

The monotonicity implies that $A(\varepsilon)$ increases (decreases) in the sense of set inclusion, which actually corresponds to Beurling's original definiton.

If $F(\rho_0, Q_0) = 0$ and A_{ρ_0} is non-degenerate, i.e., $\partial_\rho F(\rho_0, Q_0)$ is invertible, one would proceed to the successive approximation procedure as (2.2) in order to construct a solution ρ to $F(\rho, Q) = 0$ for $Q \neq Q_0$; but it fails because of the loss of derivatives $\partial_\rho F(\rho_0, Q_0)^{-1} F(\rho, Q) \in h^{2+\alpha}(\partial A_0)$ for $\rho \in h^{3+\alpha}(\partial A_0)$ as mentioned earlier. Instead, we take the alternative parabolic approach, namely, setting $Q(x,t) = Q_0(x) + tq(x) > 0$ and $\tilde{F}(\rho, t) = F(\rho, Q(t))$, we consider the evolution equation

$$\rho'(t) = -\partial_\rho \tilde{F}\left(\rho, t\right)^{-1} \left[\partial_t \tilde{F}\left(\rho, t\right)\right] \tag{2.8}$$

with $\rho(0) = \rho_0$ under the regularity condition

$$\rho \in C([0,T), h^{3+\alpha}(\partial A_0)) \cap C^1([0,T), h^{2+\alpha}(\partial A_0)). \tag{2.9}$$

In fact, this regularity assumption is suitable not only for treating loss of derivatives in (2.8), but also for applying the standard theory of evolution equations. Proposition 1 shows that, for $A(t) = A_{\rho(t)}$, (2.8) is represented by flow equation

$$\begin{aligned} & V = -\frac{p}{Q} \quad \text{on} \quad \partial A(t), \\ & \text{with} \quad \begin{cases} -\Delta p = 0 \quad \text{in} \quad \Omega \setminus \overline{A(t)}, \\ p = 0 \quad \text{on} \quad \partial\Omega, \\ \dfrac{\partial p}{\partial \nu} + \left(H + \dfrac{1}{Q}\dfrac{\partial Q}{\partial \nu}\right) p = q \quad \text{on} \quad \partial A(t), \end{cases} \end{aligned} \tag{2.10}$$

where V is the speed of moving surface $\partial A(t)$ in the outer normal direction with respect to $\Omega \setminus \overline{A(t)}$ to (1.1) for varying $Q(x,t)$. Summarizing the above argument, we obtain the following characterization of a family of solutions $A(t)$ to (1.1).

Theorem 1. *Let $Q(x,t) = Q_0(x) + tq(x) > 0$ and $Q_0, q \in h^{2+\alpha}(\mathbb{R}^n)$, and suppose that $A(0) = A_{\rho_0} \subset \Omega$ with $\rho_0 \in h^{3+\alpha}(\partial A_0)$ is a solution to* (1.1) *for Q_0. If* (2.9) *holds and $A(t) = A_{\rho(t)}$ are all non-degenerate, then the following are equivalent:*

(i) Each $A(t)$ is a solution to the Bernoulli problem (1.1) *for $Q(x,t)$;*

(ii) $\{A(t)\}_{0 \le t < T}$ is a solution to the flow equation (2.10).

Theorem 1 reduces the construction of solutions $A(t)$ of (1.1) to the solvability of the flow equation (2.10). The following theorem shows that (2.10) is, in fact, solvable locally in time.

Theorem 2. *Let Ω be a bounded domain with $h^{2+\alpha}$-boundary and $Q(x,t) = Q_0(x) + tq(x) > 0$, $Q \in h^{3+\alpha}(\mathbb{R}^n)$, and $q \in h^{2+\alpha}(\mathbb{R}^n)$, and suppose that $A_{\rho_0} \subset \Omega$ with $\rho_0 \in h^{3+\alpha}(\partial A_0)$ is a non-degenerate solution to* (1.1) *for $Q_0 > 0$.*

(i) If A_{ρ_0} is elliptic, monotone and $q < 0$, then there exists $T > 0$ such that, for all $0 \le t < T$, (1.1) *possesses a non-degenerate, elliptic and monotone solution $A(t) = A_{\rho(t)}$ for $Q(x,t)$ satisfying $\rho(0) = \rho_0$ and* (2.9).

(ii) If A_{ρ_0} is hyperbolic, monotone and $q > 0$, then there exists $T > 0$ such that, for all $0 \le t < T$, (1.1) *possesses a non-degenerate, hyperbolic and monotone solution $A(t) = A_{\rho(t)}$ for $Q(x,t)$ satisfying $\rho(0) = \rho_0$ and* (2.9).

Remark 3. We require the higher regularity $Q \in h^{3+\alpha}(\mathbb{R}^n)$, $q \in h^{2+\alpha}(\mathbb{R}^n)$ as compared to Theorem 1. This is due to the fact that we differentiate (2.8) (or (2.10)) with respect to ρ for the application of the semigroup theory.

Remark 4. Depending on the ellipticity/hyperbolicity of A_{ρ_0}, the linearized operator has the opposite sign, which reflects in the assumption $q \lessgtr 0$. Thus, in both cases (i), (ii), the moving domain $A(t)$ shrinks under the flow (2.10).

The proof of Theorem 2 is mainly based on harmonic analysis. For the functional analytic treatment of (2.10), we first derive the corresponding evolution equation in terms of ρ defined on a fixed surface ∂A_0. We then analyze the spectral properties of its linearized operator by representing its principal part as a Fourier multiplier operator, and prove that it generates a strongly continuous analytic semigroup. The details of the proof can be found in [7].

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