REPRESENTATION THEORY OF LIE SUPERALGEBRAS IN THE BGG CATEGORY

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1. Lie superalgebras

Let us denote the field of complex number by $\mathbb{C}$. In this talk all vector spaces, algebras, et cetera will be over $\mathbb{C}$.

Recall that a Lie algebra is a vector space $\mathfrak{g}$ that is equipped with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying for all $X, Y, Z \in \mathfrak{g}$:

(i) $[X, Y] = -[Y, X]$,
(ii) $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$.

(i) is called skew-symmetry and (ii) Jacobi identity.

A Lie superalgebra is a generalization of a Lie algebra. It is a vector space $\mathfrak{g}$ which is $\mathbb{Z}_2$-graded, that is, $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, equipped with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is compatible with the $\mathbb{Z}_2$-gradation, satisfying

(i) $[X, Y] = -(-1)^{|X|\cdot|Y|}[Y, X]$,
(ii) $[X, [Y, Z]] = [[X, Y], Z] + (-1)^{|X|\cdot|Y|}[Y, [X, Z]]$.

Here the notation $|X|$ denotes the $\mathbb{Z}_2$-degree of the homogeneous element $X \in \mathfrak{g}$, that is, if $X \in \mathfrak{g}_\epsilon$, then $|X| = \epsilon$, for $\epsilon = \{\bar{0}, \bar{1}\}$. For non-homogeneous elements (i) and (ii) above are extended linearly. Here (i) is called skew-supersymmetry and (ii) super Jacobi identity. An element X in $\mathfrak{g}_{\bar{0}}$ is called even, while an element $Y \in \mathfrak{g}_{\bar{1}}$ is called odd.

Remark 1.1. In particular, if $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ and $\mathfrak{g}_{\bar{1}} = 0$, then $\mathfrak{g} = \mathfrak{g}_{\bar{0}}$ is just a Lie algebra, as in this case $|X| = |Y| = 0$, and skew-supersymmetry and super Jacobi identity of a Lie superalgebra reduce to skew-symmetry and Jacobi identity for a Lie algebra.

Remark 1.2. By the previous remark we see that in general $\mathfrak{g}_{\bar{0}}$ is a Lie algebra. The super Jacobi identity in the case X and Y are even, Z odd is equivalent to the fact that the adjoint action of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is a representation of $\mathfrak{g}_{\bar{0}}$. So in general, a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ consists of Lie algebra $\mathfrak{g}_{\bar{0}}$ and a $\mathfrak{g}_{\bar{0}}$-module $\mathfrak{g}_{\bar{1}}$. Additionally, we have a $\mathfrak{g}_{\bar{0}}$-module homomorphism $S^2(\mathfrak{g}_{\bar{1}}) \to \mathfrak{g}_{\bar{0}}$ that satisfies the condition coming from super Jacobi identity with X, Y, Z all odd. In fact, a Lie superalgebra is equivalent to such a data.

We have the usual notion of subalgebra and ideals of a Lie superalgebra. A Lie superalgebra is simple, if it has no nontrivial ideals. A module M over a Lie superalgebra $\mathfrak{g}$ is always assumed to be $\mathbb{Z}_2$-graded with the action of $\mathfrak{g}$ compatible with the respective $\mathbb{Z}_2$-gradation of M. A module over a Lie superalgebra is called simple if it has no nontrivial $\mathbb{Z}_2$-graded submodules.

2. Complex simple Lie superalgebras

The first fundamental question is the classification of finite-dimensional simple Lie superalgebras. Recall that the Cartan-Killing classification of Lie algebras says that the finite-dimensional complex simple Lie algebras consist precisely of four classical infinite series denoted by A_n, B_n, C_n, and D_n, $n \geq 1$. In addition, we have five exceptional simple Lie algebras usually denoted by E_6, E_7, E_8, F_4, and G_2.

Finite-dimensional complex simple Lie superalgebras were classified by Victor Kac in 1977. A little bit earlier simple Lie superalgebras that have a non-degenerate Killing form were classified by the physicists Rittenberg, Scheunert and Nahm.

We shall explain the list of finite-dimensional simple Lie superalgebras below. For this, let us first recall how the Lie algebras of types A_n, B_n, C_n, and D_n are constructed.

The general linear Lie algebra is the Lie algebra of all linear transformation of a finite-dimensional vector space V of dimension say $n+1$, i.e., $\mathbb{C}^{n+1}$. If we chose an ordered basis for $\mathbb{C}^{n+1}$, then we can realize this Lie algebra as the Lie algebra of all $n+1 \times n+1$ complex matrices with the Lie bracket given by the usual anti-commutator, i.e., for two $n+1 \times n+1$ matrices X, Y we have $[X, Y] = XY - YX$. Then A_n is the subalgebra of this Lie algebra consisting of traceless linear transformations. It is called the special linear Lie algebra and usually denoted by $\mathfrak{sl}(V)$.

To obtain the infinite series of Lie algebras B_n, C_n, and D_n, we take a non-degenerate bilinear form on a finite-dimensional vector space and look at the Lie subalgebra of the general linear Lie algebra that preserves such a bilinear form. In the case when the bilinear form in symmetric we get the series B_n and D_n, while in the case when the bilinear form is symplectic, we get the series C_n. The orthogonal Lie algebras B_n and D_n are denoted by $\mathfrak{so}(V)$, while the symplectic Lie algebra C_n is denoted by $\mathfrak{sp}(V)$,

Now, let us take a finite-dimensional $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. Let us suppose that we have have $\dim V_{\bar{0}} = m$ and $\dim V_{\bar{1}} = n$. We call V a superspace of superdimension $(m|n)$. All superspaces of the same superdimension are isomorphic which we shall simply denote by $\mathbb{C}^{m|n}$. As V is $\mathbb{Z}_2$-graded, the space of endomorphisms $\mathrm{End}_{\mathbb{C}}(V)$ of all complex linear transformations inherits a natural $\mathbb{Z}_2$-gradation. Namely, even elements are precisely those linear transformations that preserve the degrees of V, while odd elements are precisely those that interchanges the degrees of V. The space of linear transformations then has a structure of a Lie superalgebra with bracket $[\cdot,\cdot] : \mathrm{End}_{\mathbb{C}}(V) \times \mathrm{End}_{\mathbb{C}}(V) \to \mathrm{End}_{\mathbb{C}}(V)$ defined as follows: For homogeneous linear transformations X, Y we let

$$[X, Y] := XY - (-1)^{|X|\cdot|Y|}YX.$$

This formula is then extended by linearly in the case $X, Y \in \mathrm{End}_{\mathbb{C}}(V)$. The Lie superalgebra obtained this way is called the general linear Lie superalgebra and denoted by $\mathfrak{gl}(V)$ or $\mathfrak{gl}(m|n)$, as superspaces of the same superdimension give isomorphic Lie superalgebras.

The correct generalization of trace of a linear transformation on a vector space is the notion of supertrace of a linear transformation on a superspace. If we take an ordered homogeneous basis for a superspace V of supedimension $(m|n)$ of the form $\{v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m|n}\}$ such that the elements $v_1, \ldots, v_m$ are even, and the elements $v_{m+1}, \ldots, v_{m+n}$ are odd, then we can realize any linear transformation $X : V \to V$ as an $m + n \times m + n$ matrix of the form

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A is an $m \times m$ matrix, B an $m \times n$, C an $n \times m$, and D an $n \times n$. The supertrace is defined to be

$$\mathrm{str}X = \mathrm{tr}A - \mathrm{tr}D.$$

This notion can be seen to be independent of the homogeneous ordered basis chosen, and so is well-defined.

We can now take the subalgebra of linear transformations that have zero supertrace. This turns out to be a subalgebra of $\mathfrak{gl}(m|n)$ and is simple for almost all values of $m \neq n$. In the case when $m = n$, the scalar multiple of the identity matrix, which we remark has zero supertrace, is an ideal. Dividing by this ideal we get a simple Lie algebra for $m \geq 2$. This way we obtain the super analogues of the special linear Lie algebra. These Lie superalgebras accordingly are referred to as special linear Lie superalgebras and denoted by $\mathfrak{sl}(V)$.

Now, we can take a bilinear $(\cdot|\cdot)$ for on the superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ that is even. This means that $(V_{\bar{0}}|V_{\bar{1}}) = 0$. Suppose we are given such a pairing that is supersymmetric. This means that it is symmetric when restricted to the even subspace $V_{\bar{0}}$ and skew-symmetric when restricted to the odd subspace $V_{\bar{1}}$. Assume further that it is non-degenerate. Then necessarily we have $\dim V_{\bar{1}} = 2k$ is even. Similarly to the classical setting we can consider the subalagebra of $\mathfrak{gl}(V)$ that preserves this bilinear form. It turns out that, just in the classical setting, this subalgebra is simple, and is called the ortho-symplectic Lie superalgebra and denoted by $\mathfrak{osp}(V)$ or $\mathfrak{osp}(\dim V_{\bar{0}}|\dim V_{\bar{1}})$. The name derives from the fact that the condition on the bilinear form necessarily implies that its even subalgebra contains a copy of $\mathfrak{so}(V_{\bar{0}}) \oplus \mathfrak{sp}(V_{\bar{1}})$ as a subalgebra. It turns out that this is precisely the even subalgebra of the ortho-symplectic Lie superalgebra. One can of course ask what happens if one takes a non-degenerate bilinear form that is symplectic on $V_{\bar{0}}$ and symmetric on $V_{\bar{1}}$. Such a form is referred to as super skew-symmetric. It turns out that we get Lie superalgebras that are isomorphic to the ortho-symplectic Lie superalgebras.

The above are rather straightforward generalization of simple classical Lie algebras and it gives us examples of simple Lie superalgebras. However, there are two types of finite-dimensional Lie superalgebras that have no direct analogues in finite-dimensional Lie algebras. We shall explain those below.

Recall the so-called Cartan type Lie algebras, which are infinite-dimensional Lie algebras of polynomial vector fields on a finite-dimensional complex vector space. To be more precise, let $\mathbb{C}[x_1, \ldots, x_n]$ be the ring of polynomials in n indeterminates $x_1, \ldots, x_n$. Let $\frac{\partial}{\partial x_i}$ be the usual partial derivative, i.e., it is the derivation on $\mathbb{C}[x_1, \ldots, x_n]$ uniquely determined by

$$\frac{\partial}{\partial x_i}(x_j) = \delta_{ij}.$$

The Lie algebra $W(n)$ of polynomial vector fields on the complex n-dimensional space is the Lie algebra of derivations from $\mathbb{C}[x_1, \ldots, x_n]$ to itself. Explicitly, we have

$$W(n) = \{\sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} | f_i \in \mathbb{C}[x_1, \ldots, x_n]\}.$$

It is simple, although it is clearly infinite-dimensional. Furthermore, $W(n)$ contains three series of Lie subalgebras that are all infinite-dimensional. Namely, we have $S(n)$, the subalgebra of divergence-free vector fields, i.e.,

$$S(n) = \{\sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} \in W(n) | \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(f_i) = 0\}.$$

Other series are the Hamiltonian and the contact vector fields.

Analogously, we can consider polynomial vector fields on the complex superspace of superdimension $(m|n)$. We remark that polynomials on $\mathbb{C}^{m|n}$ are $\mathbb{C}[x_1, \ldots, x_m] \otimes \wedge(\xi_1, \ldots, \xi_n)$, where $\wedge(\xi_1, \ldots, \xi_n)$ stands for the exterior algebra in the indeterminates $\xi_1, \ldots, \xi_n$. This way we obtain several series of simple Lie superalgebras. They are in general infinite-dimensional. However, when $m = 0$, we get finite-dimensional Lie superalgebras. Indeed, we obtain four series of finite-dimensional simple Lie superalgebras this way. They are super analogues of the W-series, S-series and the Hamiltonian series, in addition to a deformation of the S-series.

The other type of Lie superalgebras with no classical analogues are called the strange types. To describe them, consider a superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ such that $\dim V_{\bar{0}} = \dim V_{\bar{1}} = n$. In this case, there exists an involution of V which swaps $V_{\bar{0}}$ and $V_{\bar{1}}$. We can consider the Lie subalgebra of $\mathfrak{gl}(V)$ consisting of linear tranformations that commute with this involution. We get a Lie superalgebra $\mathfrak{q}(n)$ called the queer Lie superalgebra. To get a simple Lie superalgebra we first take the derived subalgebra and then mod out by the scalar linear transformations. The resulting Lie superalgebra is simple for for $n \geq 3$. Another thing that we can do when $\dim V_{\bar{0}} = \dim V_{\bar{1}} = n$ is to take a symmetric non-degenerate bilinear form $(\cdot|\cdot)$ such that $(V_{\bar{0}}|V_{\bar{0}}) = (V_{\bar{1}}|V_{\bar{1}}) = 0$. This means of course that $V_{\bar{0}}$ is paired with $V_{\bar{1}}$ non-degenerately. We can take the subalgebra of $\mathfrak{gl}(V)$ that preserves this form, and get a Lie superalgebra called the periplectic Lie superalgebra and usually denoted by $\mathfrak{p}(n)$. It turns out that the derived subalgebra is simple for $n \geq 3$.

The above explains all infinite series of finite-dimensional Lie superalgebras over $\mathbb{C}$. They include all the classical finite-dimensional Lie algebras as well. However, the list does not include the exceptional Lie algebras. So to have a complete list we need to include E_6, E_7, E_8, F_4, and G_2 as well. It turns out that there are three exceptional Lie superalgebras that are NOT Lie algebras. They are $F(3|1)$, $G(3)$, and the one-parameter family $D(2|1,\zeta)$, where $\zeta \in \mathbb{C}\setminus\{0,-1\}$. The existence of these three exceptional Lie superalgebras were suggested by Freund and Kaplansky.

Let us describe these exceptional Lie superalgebras. As we have mentioned earlier, if $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a Lie superalgebra, then $\mathfrak{g}_{\bar{0}}$ is a Lie algebra and $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module under the adjoint action. For these three exceptional Lie superalgebras we have:

$$\begin{aligned}
&D(2|1,\zeta)_{\bar{0}} \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2, \quad D(2|1,\zeta)_{\bar{1}} \cong \mathbb{C}^2 \boxtimes \mathbb{C}^2 \boxtimes \mathbb{C}^2.\\
&G(3)_{\bar{0}} \cong G_2 \oplus \mathfrak{sl}_2, \quad G(3)_{\bar{1}} \cong \mathbb{C}^7 \boxtimes \mathbb{C}^2.\\
&F(3|1)_{\bar{0}} \cong \mathfrak{so}_7 \oplus \mathfrak{sl}_2, \quad F(3|1)_{\bar{1}} \cong \mathrm{spin}_7 \boxtimes \mathbb{C}^2.
\end{aligned}$$

Above $\mathbb{C}^2$ denotes the natural $\mathfrak{sl}_2$-module, $\mathbb{C}^7$ denotes the smallest non-trivial module of the exceptional Lie algebra G_2, and spin_7 is the (8-dimensional) spin module of the Lie algebra $\mathfrak{so}_7$. We mention here that we have isomorphisms:

$$D(2|1,\zeta) \cong D(2|1,\frac{1}{\zeta}) \cong D(2|1,-1-\zeta).$$

Also, $D(2|1,1) \cong \mathfrak{osp}(4|2)$, so that $D(2|1,\zeta)$ may be regarded as a deformation of $\mathfrak{osp}(4|2)$.

3. Representation Theory of Lie superalgebras

As for Lie algebras, modules over Lie superalgebras are the same as representations of Lie superalgebras.

3.1. Finite-dimensional module category of simple Lie algebras have been studied early on. There are two fundamental questions.

(i) Classification of finite-dimensional irreducible modules.
(ii) Computation of the characters for these irreducible modules.

Similar to the classical representation theory of finite-dimensional simple Lie algebras the simple Lie superalgebras have triangular decomposition. So it makes sense to talk about highest weights with respect to a Borel subalgebra. It is easy to see that every finite-dimensional irreducible module over a simple Lie superalgebra is necessarily a highest weight module. However, certainly not every highest weight irreducible module is finite-dimensional.

Question (i) asks to determine these highest weights with respect to a Borel subalgebra that are highest weights of finite-dimensional irreducible modules. This question turns out to be not so difficult and it was settled quite early on in Kac's classical paper in which he obtained the classification of finite-dimensional simple Lie superalgebras.

Question (ii) is asking to compute the characters of all finite-dimensional irreducible module over a complex simple Lie superalgebra. We note that, contrary to theory of semisimple Lie algebras, the finite-dimensional representations of finite-dimensional simple Lie superalgebras are not completely reducible. Only for the Lie superalgebra $\mathfrak{osp}(1|2n)$ are all finite-dimensional representations completely reducible. In any case, this problem is much harder, and only by now, has the problem been settled completely. We mention the contributions by Kac (for so-called typical modules), Serganova (for $\mathfrak{gl}(m|n)$), Penkov-Serganova (for $\mathfrak{q}(n)$), Gruson-Serganova (for $\mathfrak{osp}$), Sergeev, van der Jeugt, Brundan (new conceptual viewpoint), Germoni and many others. Indeed, only recently the finite-dimensional irreducible characters for $\mathfrak{p}(n)$ has been computed by Balagovic and 9 others (including Serganova), which completes the irreducible character problem for finite-dimensional modules.

3.2. Computations of the irreducible characters of various Lie superalgebras in the Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$ were done in the last decade or so. Here the first fundamental work is a conjecture by Brundan in 2003 on the irreducible characters of the Lie superalgebra $\mathfrak{gl}(m|n)$ in category $\mathcal{O}$. We shall describe this conjecture in some detail below.

Let $\mathfrak{gl}_\infty$ be the Lie algebra consisting of matrices with rows and columns indexed by $\mathbb{Z}$ with finitely many nonzero entries. Let $\mathcal{U}_q$ be the associated quantum group over $\mathbb{Q}(q)$. Let $\mathbb{V}$ be its standard module with basis $\{v_i|i \in \mathbb{Z}\}$ and $\mathbb{V}^*$ be its restricted dual with normalized dual basis $\{w_j|j \in \mathbb{Z}\}$. In the terms of Lusztig $\mathbb{V}$ and $\mathbb{V}^*$ are so-called based modules, i.e., they have standard monomial bases and distinguished bases that he referred to as canonical and dual canonical bases. (In language of Kashiwara they are called lower and upper global bases.) Now using the quasi-$\mathcal{R}$-matrix Lusztig showed that tensor product of based modules are based modules. So in particular, the U_q-module

$$\mathbb{T}^{m|n} = \underbrace{\mathbb{V} \otimes \cdots \otimes \mathbb{V}}_{m} \otimes \underbrace{\mathbb{V}^* \otimes \cdots \otimes \mathbb{V}^*}_{n}$$

has standard monomial of the form

$$M_f := v_{f(1)} \otimes \cdots \otimes v_{f(m)} \otimes w_{f(m+1)} \otimes \cdots \otimes w_{f(m+n)},$$

where $f:\{1,\cdots,m+n\}\to\mathbb{Z}$, and canonical and dual canonical bases, which we denote by T_f and L_f. We should mention that formulas for the (dual) canonical bases are in general complicated.

Let X denote the set of integral weights. For an integral weight λ we associate a function $f_\lambda:\{1,\ldots,m+n\}\to\mathbb{Z}$ by $f_\lambda(i):=(\lambda,\delta_i)$, where δ_i are the dual to the standard basis of the Cartan subalgebra of $\mathfrak{gl}(m|n)$. This gives a bijection between X and the set of $\mathbb{Z}$-valued functions on $\{1,\ldots,m+n\}$.

Let $K(\mathcal{O}^{m|n})$ be the split Grothendieck group of the BGG category of integral weight modules over $\mathfrak{gl}(m|n)$. Then $K(\mathcal{O}^{m|n})$ has several distinguished bases. The first set of basis consists of the isomorphism classes of Verma modules $\{[\Delta(\lambda)]|\lambda\in X\}$. Other bases are the isomorphism classes of irreducible modules $\{[L(\lambda)]|\lambda\in X\}$. Another set of bases consists od the isomorphism classes of tilting modules $\{[T(\lambda)]|\lambda\in X\}$. Above $Y(\lambda)$ denotes the corresponding module of highest weight $\lambda-\rho$, where $Y=\Delta,T,L$ and ρ is the usual half super sum of positive roots of $\mathfrak{gl}(m|n)$.

The conjecture of Brundan says that the linear isomorphism from $\Psi:K(\mathcal{O}^{m|n})\otimes_{\mathbb{Z}}\mathbb{Q}\to\mathbb{T}^{m|n}_{q=1}$ (evaluated at $q=1$) defined by $\Psi([\Delta(\lambda)]):=M_{f_\lambda}$ sends $[T(\lambda)]$ to T_{f_λ} and $[L(\lambda)]$ to L_{f_λ}.

As the character of Verma modules are easily computed, the problem of computing the character of $L(\lambda)$ is reduced to the computation of the transition matrix between the two bases $\{M_f\}$ and $\{L_f\}$. Actually, this problem is equivalent to the problem of finding the transition matrix between the bases $\{M_f\}$ and $\{T_f\}$. In any case, there are algorithms that can compute the coefficients of these matrices, and hence the problem of computing the irreducible character of $\mathfrak{gl}(m|n)$ in the BGG category of integral weight modules is reduced to proving the conjecture. The remarkable thing about the conjecture is that it would imply that canonical basis of classical Lie algebras plays a fundamental role in the representation theory of Lie superalgebras. The conjecture was proved in 2015 by Lam, Wang and myself. The proof uses two important ingredients. The first is a well-known formulation of the Kazhdan-Lusztig conjecture for type A Lie algebras via the Schur-Weyl-Jimbo duality. The second important ingredients is the concept of super duality which is an equivalence of certain parabolic subcategories between Lie algebra representations and Lie superalgebra representations. Based on these we were able to reduce the conjecture to the classical Kazhdan-Lusztig conjecture for semisimple Lie algebras which was proved by independently by Brylinski-Kashiwara and Bernstein-Beilinson. This solves the irreducible character problem in the BGG category for the type $\mathfrak{gl}$ Lie superalgebras.

Recent works by Bao and Wang, and Bao settled that irreducible character problem for $\mathfrak{osp}$-type Lie superalgebras. The idea follows the idea of the $\mathfrak{gl}$-type paper of Cheng-Lam-Wang. There are two ingredients needed, the first is a formulation of the Kazhdan-Lusztig theory of type BCD Lie algebras in terms of "canonical" bases. The second ingredient is already present, namely super duality, which appeared in its first rudimentary form in 2004 in a joint paper with Wang, and Zhang. In more general forms it appears in joint works with Lam and Wang, and finally in its most general form in a joint work with Kwon and Wang. The problem is then to find some kind of standard and canonical bases whose transition matrices are precisely the classical Kazhdan-Lusztig polynomials of type BD. It turns out that one cannot uses quantum groups anymore. The solution that Bao and Wang present is to use a quantum deformation of symmetric pairs, called quantum symmetric pairs. They develop a theory parallel to the theory of canonical basis, which they referred to as ι-canonical bases and proved that they indeed recover the classical Kazhdan-Lusztig polynomials of type BD as coefficients of the transition matrices between standard and ι-canonical basis. With the theory of ι-canonical basis and super duality at their disposal, Bao and Wang solved the irreducible character problem for type B Lie superalgebras. This work lays the foundation for much further development of quantum symmetric pair. Type D is then solved by Bao in a later work. In any case, this completely settles the irreducible character problem in the BGG category for the type $\mathfrak{osp}$ Lie superalgebras.

For $\mathfrak{q}(n)$ the problem has not been completely settled yet at this point. However, because of its remarkable connection to classical (non-super) Lie theory, we shall briefly explain what is known.

Here, we shall consider first the full BGG category, that is, including modules with all possible highest weights, not just integral highest weights. In his thesis Chih-Whi Chen showed that the computation of the irreducible character of a module of any highest weight can be reduced to the calculation of three types of highest weights:

(A) congruent s-type weights, where $s \notin \frac{1}{2}\mathbb{Z}$.
(B) integral weights,
(C) half-integer weights,

Remark 3.1. We remark that the same problem for the Lie superalgebra $\mathfrak{gl}(m|n)$ is easy in the sense that no matter what the highest weight is, the problem can be reduced to the integral weight case. This was proved in a joint work with Mazorchuk and Wang in 2014. So Brundan's conjecture settles all cases.

Let us consider the case (A), i.e., the type s-weights with $s \notin \frac{1}{2}\mathbb{Z}$. It was conjectured in a joint work with Kwon and Wang that the irreducible characters are given by the same Kazhdan-Lusztig polynomials as for type $\mathfrak{gl}$ Lie superalgebras. Thus, the canonical basis of a certain Fock space over the quantum group of type A Lie algebra gives the solution. This was indeed proved by Brundan and Davidson in 2017. A special case was established earlier in a joint work with Chih-Whi Chen.

Let us describe the case (C) first. Studying the linkage in the category $\mathcal{O}$, it is plausible to conjecture that we can replace the type A quantum group in case (A) by the type C quantum group and take the Fock space to be the tensor power of the standard module. This space has standard, canonical and dual canonical bases and it would natural to conjecture that they would give the irreducible characters in this case. However, computer calculation by Tsuchioka shows that these canonical basis lack positivity and so such a conjecture cannot be true. However, if one can replace Lusztig's canonical basis by Webster "orthodox" basis, which are related to some projective modules over certain algebra so that positivity would always hold. It was conjectured in the same joint work with Kwon and Wang that indeed Webster's basis should solve the irreducible character problem in this case. Indeed, this was proved again by Brundan and Davidson in 2017.

Finally, the case (B) is still open at this moment. In 2004 Brundan conjectured that the canonical basis on a certain natural module over the type B quantum group should give the irreducible character in this case. However, Tsuchioka's computer calculation shows that the conjecture cannot be true because of lack of positivity of the corresponding canonical basis. Nevertheless, from various viewpoints, it would reasonable to believe that the quantum group associated to the Lie superalgebra $\mathfrak{osp}(1|2n)$, with $n \to \infty$, should play a similar role here as in the case (C).

Now, we shall report on recent progress in the study of irreducible characters for the exceptional Lie superalgebras $D(2|1,\zeta)$ and $G(3)$ in the BGG category $\mathcal{O}$. This is based on joint works with Weiqiang Wang. The problem for the Lie superalgebra $F(3|1)$ is still open at this moment.

We shall explain the strategy, as it would be very tedious to give all the formulas and details here, as in the case of exceptional Lie superalgebras one cannot expect to have solutions of closed form. Again, computing the irreducible characters in the category $\mathcal{O}$ is equivalent to computing the characters of the tilting modules. We shall continue to use the notation $\Delta(\lambda)$, $L(\lambda)$, $T(\lambda)$ for Verma, irreducible, and tilting modules of highest weight $\lambda - \rho$, respectively. As $T(\lambda)$ has a Δ-flag we have the following identity in the Grothendieck group:

$$[T(\lambda)] = \sum_{\mu} b_{\mu\lambda}[\Delta(\mu)],$$

where $b_{\mu\lambda} \in \mathbb{N}$. So, we need to compute $b_{\mu\lambda}$.

A weight ν is called typical if $(\nu, \beta) \neq 0$, for all isotropic roots β. Now a typical block is equivalent to certain blocks of $\mathfrak{g}_{\bar{0}}$-modules by a result of Gorelik. Thus, the characters of tilting modules of typical highest weights are known to be given by classical Kazhdan-Lusztig polynomials.

In general let E be a finite-dimensional $\mathfrak{g}$-modules and consider $E \otimes T(\nu)$. In general $E \otimes T(\nu)$ is a direct sum of $\mathfrak{g}$-modules each lying in different blocks. We can thus project $E \otimes T(\nu)$ into a fixed block determined by a fixed central character. Let us denote such a projection by $\mathcal{E}T(\nu)$, and we may regard $\mathcal{E}$ as a functor, called a translation functor. It is fairly straightforward to show that $\mathcal{E}T(\nu)$ is a direct sum of tilting modules.

Remark 3.2. It is very straightforward to compute $[\mathcal{E}T(\nu)]$ once we know the character of $T(\nu)$. Here we can use Gorelik's results as an initial step. That is, we start with a typical highest weight ν for which the character of $T(\nu)$ is known by Gorelik and apply a translation functor to it so that $\mathcal{E}T(\nu)$ is known.

Now, suppose λ is a weight for which we want to find the character of $T(\lambda)$. Suppose we can choose a finite-dimensional module E and a weight ν in such a way that $\mathrm{ch}T(\nu)$ is known and

$$[\mathcal{E}T(\nu)] = [\Delta(\lambda)] + \sum_{\mu<\lambda} b_\mu[\Delta(\mu)].$$

Suppose that we can show that $\mathcal{E}T(\nu)$ is indecomposable. Then it follows from the characterization of tilting modules that $\mathcal{E}T(\nu) = T(\lambda)$. Thus, we have computed the character of $T(\lambda)$, which is what we want. We remark that this is essentially our strategy to solve the irreducible character problem for $D(2|1,\zeta)$ and $G(3)$. There are two points we wish to emphasize:

- The calculation of $[\mathcal{E}T(\nu)]$ is straightforward, but since the Δ-flag can be quite long it is very difficult and tedious to calculate by hand. Here, we are aided by Mathematica.
- The technically most challenging part is to deal with the issue of indecomposability of $\mathcal{E}T(\nu)$. Different translation functors result in modules of different Δ-length. Here we also first employ Mathematica to find a translation functor that would result in a module with an as short a Δ-flag as possible. After that we then attempt to check indecomposability. Most of the time, it is indecomposable, however, in several instances $\mathcal{E}T(\nu)$ is decompsable. In such cases we need to resort to different methods case by case.

To conclude we obtain explicit character formulas for all tilting modules for $D(2|1,\zeta)$, for all $\zeta \in \mathbb{C} \setminus \{0,-1\}$, and for all but one single tilting modules for $G(3)$.

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