特別講演

Geometric estimates arising in the analysis of Zakharov systems

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1. Motivation

The Cauchy problem

$$\begin{aligned} i\partial_t u + \Delta u &= nu \\ \Box n &= \Delta |u|^2 \end{aligned}$$

for $u : \mathbb{R}^{d+1} \to \mathbb{C}$ and $n : \mathbb{R}^{d+1} \to \mathbb{R}$ with initial data

$$(u(0), n(0), \partial_t n(0)) = (u_0, n_0, n_1)$$

arises as a model in plasma physics and was formulated by Zakharov in [23]. Here, Δ denotes the Laplacian on $\mathbb{R}^d$ and $\Box = \partial_t^2 - \Delta$ is the D'Alembertian on $\mathbb{R}^{d+1}$. Numerous contributions have been made towards understanding the wellposedness of this system, including papers by Sulum–Sulum [22], Ozawa–Tsutsumi [21], Bourgain–Colliander [14], Ginibre–Tsutsumi–Velo [18], Bejenaru–Herr–Holmer–Tataru [5], and Bejenaru–Herr [4]. We refer the reader to the papers cited above for further background on the physical relevance of the above Zakharov system, and for a more comprehensive overview of the existing literature on this system.

It is natural to consider initial data (u_0, n_0, n_1) in the spaces

$$H^k(\mathbb{R}^d) \times H^\ell(\mathbb{R}^d) \times H^{\ell-1}(\mathbb{R}^d),$$

where $H^k(\mathbb{R}^d)$ denotes the L^2-based Sobolev space of order k, and to determine which pairs (k, ℓ) give rise to local wellposedness. In this context, there is a sensible notion of criticality for the Zakharov system and, for $d \geq 4$, a local wellposedness result was established in [18] in the whole subcritical range of (k, ℓ). The case where $d \leq 3$ is more problematic, and here we focus on the more recent results obtained in [5] and [4] in dimensions $d = 2$ and $d = 3$, respectively, which successfully closed the full subcritical regime in these dimensions. Certain geometric estimates were key to the breakthroughs in [5] and [4] and such estimates will be the focus of this talk.

Very roughly speaking, the approach in [5] and [4] is to employ a frequently used iteration argument involving Bourgain spaces adapted to the Schrödinger operator $i\partial_t + \Delta$ and the half-wave operators $i\partial_t \pm |\nabla|$. The crucial multilinear estimates on which the iteration is based give appropriate control on products of functions which are localised in frequency and modulation. A particular scenario (transverse high-high interactions, low modulation) reduces to certain singular convolution estimates which we describe in the next section. Before that, we note that a similar approach has been taken in more recent papers by Kinoshita [20] for the Klein–Gordon–Zakharov system in two dimensions, and Hirayama–Kinoshita [19] for a system of quadratic derivative nonlinear Schrödinger equations.

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2. Singular convolution estimates

For $j = 1, \ldots, m$, let $S_j \subset \mathbb{R}^d$ be a d_j-dimensional, compact and sufficiently smooth submanifold. We parametrise S_j by

$$\Sigma_j : U_j \subset \mathbb{R}^{d_j} \to \mathbb{R}^d$$

and let σ_j be the associated measure on S_j given by

$$\int_{\mathbb{R}^d} \varphi \, \mathrm{d}\sigma_j = \int_{U_j} \varphi(\Sigma_j(y)) \, \mathrm{d}y.$$

Given appropriate geometric assumptions on the submanifolds S_j, it is natural to seek a characterisation of exponents for which the singular convolution estimates

$$\|g_1 \mathrm{d}\sigma_1 * \cdots * g_m \mathrm{d}\sigma_m\|_{L^q(\mathbb{R}^d)} \lesssim \prod_{j=1}^{m} \|g_j\|_{L^{p_j}(\mathrm{d}\sigma_j)} \tag{1}$$

hold. Before moving forward in a somewhat general setup, we consider the following case in $\mathbb{R}^3$ which is relevant to the argument in [5] in the analysis of the Zakharov system.

Consider S_1, S_2, S_3 as pieces of the coordinate hyperplanes

$$S_j = \{w \in [-1,1]^3 : w_j = 0\}$$

parametrised by $\Sigma_j : Q = [-1,1]^2 \to S_j$ with

$$\begin{aligned} \Sigma_1(x_1, x_2) &= (0, x_1, x_2) \\ \Sigma_2(y_1, y_2) &= (y_1, 0, y_2) \\ \Sigma_3(z_1, z_2) &= (z_1, z_2, 0). \end{aligned}$$

Then

$$g_1 \mathrm{d}\sigma_1 * g_2 \mathrm{d}\sigma_2 * g_3 \mathrm{d}\sigma_3(0) = \int_{Q \times Q \times Q} f_1(x) f_2(y) f_3(z) \, \delta(F(x,y,z)) \, \mathrm{d}x \mathrm{d}y \mathrm{d}z$$

where $f_j := g_j \circ \Sigma_j$ and $F : Q \times Q \times Q \subset \mathbb{R}^6 \to \mathbb{R}^3$ is given by

$$\begin{aligned} F(x,y,z) &= \Sigma_1(x) + \Sigma_2(y) + \Sigma_3(z) \\ &= (y_1 + z_1, x_1 + z_2, x_2 + y_2) \end{aligned}$$

and therefore

$$g_1 \mathrm{d}\sigma_1 * g_2 \mathrm{d}\sigma_2 * g_3 \mathrm{d}\sigma_3(0) = \int_{[-1,1]^3} f_1(x_1, x_2) f_2(y_1, -x_2) f_3(-y_1, -x_1) \, \mathrm{d}y_1 \mathrm{d}x_1 \mathrm{d}x_2.$$

It now follows from the classical Loomis–Whitney inequality (which is a special case of the Brascamp–Lieb inequality – see (5) below) that

$$|g_1 \mathrm{d}\sigma_1 * g_2 \mathrm{d}\sigma_2 * g_3 \mathrm{d}\sigma_3(0)| \leq \prod_{j=1}^{3} \|f_j\|_{L^2(Q)} = \prod_{j=1}^{3} \|g_j\|_{L^2(\mathrm{d}\sigma_j)}. \tag{2}$$

By duality, this fact has the rather appealing geometric interpretation given by Bejenaru–Herr–Tataru in [6]. Indeed, it follows from (2) and duality that

$$\|g_1 \mathrm{d}\sigma_1 * g_2 \mathrm{d}\sigma_2\|_{L^2(\mathrm{d}\sigma_3)} \leq \|g_1\|_{L^2(\mathrm{d}\sigma_1)} \|g_2\|_{L^2(\mathrm{d}\sigma_2)}$$

which tells us that the convolution of two L^2 functions supported each on S_1 and S_2 has a well-defined restriction, as an L^2 function, to the third hyperplane S_3.

For the application to the Zakharov system, it was important to establish a *nonlinear* generalisation of the above fact in the sense that S_1, S_2 and S_3 are general compact and sufficiently smooth hypersurfaces in $\mathbb{R}^3$ which are *transversal*. Here, by transversal we mean that if one takes any collection n_1, n_2 and n_3, where n_j is a normal vector to S_j, then $\{n_1, n_2, n_3\}$ is linearly independent. The extension of (2) to hypersurfaces which are not necessarily flat is highly non-trivial and, not surprisingly, one is led to try and establish a nonlinear generalisation of the classical Loomis–Whitney inequality.

3. Brascamp–Lieb inequalities

3.1. The classical case

We begin by introducing the celebrated Brascamp–Lieb inequality. The inequality has the form

$$\int_{\mathbb{R}^d} \prod_{j=1}^{m} (f_j \circ L_j)^{c_j} \leq C \prod_{j=1}^{m} \Big(\int_{\mathbb{R}^{d_j}} f_j \Big)^{c_j} \tag{3}$$

where $m, d, d_j \in \mathbb{N}$, the $L_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ are surjective linear transformations, $0 \leq c_j \leq 1$ and we consider nonnegative integrable functions f_j on $\mathbb{R}^{d_j}$. Clearly, an equivalent formulation is

$$\Big| \int_{\mathbb{R}^d} \prod_{j=1}^{m} f_j \circ L_j \Big| \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^{d_j})} \tag{4}$$

where $f_j \in L^{p_j}(\mathbb{R}^{d_j})$ and $p_j = c_j^{-1} \geq 1$ for each j.

As in the [11] we denote by $\mathrm{BL}(\mathbf{L}, \mathbf{c})$ the smallest constant C for which (3) holds for all nonnegative input functions $f_j \in L^1(\mathbb{R}^{d_j})$, $1 \leq j \leq m$, where $\mathbf{L} = (L_j)_{j=1}^m$ and $\mathbf{c} = (c_j)_{j=1}^m$. That is,

$$\mathrm{BL}(\mathbf{L}, \mathbf{c}) = \sup \frac{\int_{\mathbb{R}^d} \prod_{j=1}^m (f_j \circ L_j)^{c_j}}{\prod_{j=1}^m (\int_{\mathbb{R}^{d_j}} f_j)^{c_j}}$$

where the supremum is taken over all $\mathbf{f} = (f_j)_{j=1}^m$ for which $0 < \int_{\mathbb{R}^{d_j}} f_j < \infty$. Also, we call $(\mathbf{L}, \mathbf{c})$ the *Brascamp–Lieb datum*, and $\mathrm{BL}(\mathbf{L}, \mathbf{c})$ the *Brascamp–Lieb constant*.

Important special cases of Brascamp–Lieb datum include the case where each $B_j : \mathbb{R}^d \to \mathbb{R}^d$ is the identity transformation and $\sum_{j=1}^m c_j = 1$, in which case $\mathrm{BL}(\mathbf{L}, \mathbf{c}) = 1$ is the multilinear Hölder inequality. Also, Young's convolution inequality in dual form (see (9) below) clearly fits into the above framework and it was in Brascamp and Lieb's famous paper [15] on the sharp constant in Young's convolution inequality that the systematic study of (3) originates.

Most relevant to the preceding discussion on the Zakharov system are Loomis–Whitney-type estimates. The classical version states that

$$\int_{\mathbb{R}^d} \prod_{j=1}^{d} (f_j \circ \pi_j)^{\frac{1}{d-1}} \leq \prod_{j=1}^{d} \Big(\int_{\mathbb{R}^{d-1}} f_j \Big)^{\frac{1}{d-1}} \tag{5}$$

for nonnegative integrable functions f_j on $\mathbb{R}^{d-1}$, where $\pi_j(x)$ omits x_j; that is

$$\pi_1(x) = (x_2, \dots, x_d)$$

and so on. A "block form" generalisation of this inequality, where the L_j have higher dimensional kernels, is also valid. If $L_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ satisfy

$$\bigoplus_{j=1}^{m} \ker L_j = \mathbb{R}^d \tag{6}$$

and $c_j = \frac{1}{m-1}$ for all $j = 1, \dots, m$, then $\mathrm{BL}(\mathbf{L}, \mathbf{c}) = 1$.

Naturally, it is desirable to seek necessary and conditions for the finiteness of the Brascamp–Lieb constant. The first result of this type in full generality was obtained in [11] (see [1] and [16] in the case of rank-1 maps L_j) where it was shown that

$$\mathrm{BL}(\mathbf{L}, \mathbf{c}) < \infty \qquad \Leftrightarrow \qquad (7) \text{ and } (8)$$

where (7) is the scaling condition

$$d = \sum_{j=1}^{m} c_j d_j \tag{7}$$

and (8) is the condition

$$\dim(V) \leq \sum_{j=1}^{m} c_j \dim(L_j V) \qquad \text{for all subspaces } V \text{ of } \mathbb{R}^d. \tag{8}$$

Observe that although (8) must be tested on all subspaces of $\mathbb{R}^d$, it consists of only *finitely* many linear inequalities in the c_j and therefore gives rise to the *finiteness polytope*

$$\Pi(\mathbf{L}) = \{\mathbf{c} : \mathrm{BL}(\mathbf{L}, \mathbf{c}) < \infty\}.$$

Following [11], we refer to a Brascamp–Lieb datum $(\mathbf{L}, \mathbf{c})$ as *simple* if $\mathbf{c}$ lies on the interior of $\Pi(\mathbf{L})$. For example, the sharp version of Young's convolution inequality on $\mathbb{R}^d$ (due to Beckner [3], [2] and Brascamp–Lieb [15]) in dual form is

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f_1(y)^{c_1} f_2(x-y)^{c_2} f_3(x)^{c_3} dx dy \leq C \prod_{j=1}^{3} \left(\int_{\mathbb{R}^d} f_j \right)^{c_j} \tag{9}$$

where $c_j \in [0,1]$, $\sum_{j=1}^{3} c_j = 2$, and the sharp constant is given by

$$C = \left(\prod_{j=1}^{3} \frac{(1-c_j)^{1-c_j}}{c_j^{c_j}} \right)^{d/2}. \tag{10}$$

In this setup, simple data corresponds to the case where $c_j \in (0,1)$ for $j = 1, 2, 3$, in which case the sharp constant satisfies $C < 1$. Moreover, the sharp constant is attained by appropriate gaussian inputs $\mathbf{f}$.

3.2. The nonlinear case

As mentioned above already, nonlinear generalisations of the Brascamp–Lieb inequality have naturally arisen in the study of the Zakharov system ([5], [6], [4]), whereby we wish to relax the assumption that the mappings L_j are linear. That is, we would like to replace the linear surjections L_j by smooth submersions B_j which are locally defined in a neighbourhood of a point $x_0 \in \mathbb{R}^d$, and see whether the inequality remains true and in what sense. The particular case of nonlinear Loomis–Whitney was first studied slightly earlier by Bennett, Carbery and Wright [13] and their motivation came from applications to multilinear restriction theory for the Fourier transform.

3.2.1. Nonlinear Loomis–Whitney-type inequalities

We begin a brief overview of the development of the theory of nonlinear Brascamp–Lieb inequalities by focussing on the case of Loomis–Whitney-type inequalities; in particular, nonlinear generalisations of the fact noted above that $\mathrm{BL}(\mathbf{L}, \mathbf{c}) = 1$ when the linear surjections $L_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ satisfy the transversality condition (6) and $c_j = \frac{1}{m-1}$ for all $j = 1, \ldots, m$.

Theorem 3.1 ([13],[6],[7]). *Let $\kappa, \nu > 0$. Suppose that $B_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$ is a C^2 submersion which satisfies $\|B_j\|_{C^2} \leq \kappa$ in a neighbourhood of $x_0 \in \mathbb{R}^d$. If $\mathrm{d}\mathbf{B}(x_0) = (\mathrm{d}B_j(x_0))_{j=1}^m$ satisfies* (6) *and*

$$\left| \star \bigwedge_{j=1}^{m} \star X_j(\mathrm{d}B_j(x_0)) \right| \geq \nu \tag{11}$$

then there exists a neighbourhood U of x_0, depending only on κ, ν and d, and a constant C, depending only on d, such that

$$\int_U \prod_{j=1}^{m} (f_j \circ B_j)^{c_j} \leq \frac{C}{\nu^{\frac{1}{m-1}}} \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{c_j}.$$

Here, for a given linear map $L_j : \mathbb{R}^d \to \mathbb{R}^{d_j}$, we are writing $X_j(L_j)$ for the wedge product of the rows of the $d_j \times d$ matrix L_j, and $\star$ denotes the Hodge star operator acting on the appropriate exterior algebra. Assumption (11) is a (quantitative) transversality condition, and in the case of the classical Loomis–Whitney data where the L_j are the mappings π_j above with one-dimensional kernels, then condition (11) is equivalent to a lower bound on the determinant of the d by d matrix with $X_j(\mathrm{d}B_j(x_0)) \in \mathbb{R}^d$, $j = 1, \ldots, d$, as the columns.

In [13], Theorem 3.1 was established in the case where $d_j = d - 1$ for each $j = 1, \ldots, d$, corresponding to the classical Loomis–Whitney inequality, and the proof was based on the so-called "method of refinements" of Christ [17]. This result was applied to establish a trilinear restriction inequality in $\mathbb{R}^3$ (specifically, the case $d = 3$ in Theorem 3.2 below). A few years later, for $d = 3$, a proof based on "induction-on-scales" was given in [6] and, moreover, for the application to the Zakharov system in two dimensions in [5], a more suitable quantitative version was established in [6].

The contribution in [7] was to take the induction-on-scales approach forward and extend to the case of Brascamp–Lieb data satisfying (6). Moreover, the following multilinear restriction estimate was deduced. Associated to the parametrisation $\Sigma_j : U_j \subset \mathbb{R}^{d-1} \to \mathbb{R}^d$ of the $(d-1)$-dimensional smooth hypersurface $S_j \subset \mathbb{R}^d$ is the *extension* (or, *adjoint restriction*) operator E_j given by

$$E_j g(\xi) = \int_{U_j} g(x) e^{i\Sigma_j(x)\cdot\xi}\,\mathrm{d}x.$$

We shall say that $S_1,\dots,S_d$ are *transversal* if $\{n_1,\dots,n_d\}$ is linearly independent, where n_j is any normal to S_j.

Theorem 3.2 ([13],[7]). *Let $d \geq 3$ and suppose that $S_1,\dots,S_d$ are transversal in a neighbourhood of the origin. Then*

$$\Big\| \prod_{j=1}^{d} E_j g_j \Big\|_{L^q(\mathbb{R}^d)} \lesssim \prod_{j=1}^{d} \|g_j\|_{L^p(U_j)} \tag{12}$$

for all g_j supported in a sufficiently small neighbourhood of the origin. Here, $(p,q) = (\frac{2d-2}{2d-3}, 2)$.

We note that a much wider range of multilinear restriction estimates of the type (12) (with more general (p,q)) are known and typically are derived in a slightly weaker form where the norm on the left-hand side is taken over a large ball B_R and the factor R^ε appears on the left-hand side; see, for example, the groundbreaking work of Bennett, Carbery and Tao [12]. Also, it is not essential to restrict attention to hypersurfaces and extensions to more general submanifolds are possible; see, for example, [4], [10], [9] and [24]. We note that the estimates in [4] were stated in the equivalent convolution form (1) (via Plancherel's theorem) and were applied to the three-dimensional Zakharov system.

3.2.2. General nonlinear Brascamp–Lieb inequalities

Very recently, we have significantly extended Theorem 3.1 to include nonlinear versions of any Brascamp–Lieb inequality with simple data. Moreover, we obtain such an estimate with a constant which is as tight as it can possibly be.

Theorem 3.3 ([8]). *Let $\varepsilon > 0$. If the datum $(\mathbf{L},\mathbf{c})$ is simple, and $\mathbf{B}$ is a family of C^2 submersions in a neighbourhood of a point $x_0 \in \mathbb{R}^d$ for which $dB_j(x_0) = L_j$ for each $j = 1,\dots,m$, then there exists a neighbourhood U of x_0 for which*

$$\int_U \prod_{j=1}^{m} (f_j \circ B_j)^{c_j} \leq (1+\varepsilon)\mathrm{BL}(\mathbf{L},\mathbf{c}) \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{d_j}} f_j \right)^{c_j}.$$

We are hopeful that rather general nonlinear Brascamp–Lieb inequalities of this type will find applications in the mathematical theory of PDE, possibly in the spirit of [5], [4], [20], [19]. Looking out more widely, given the vast applications of the classical form of the Brascamp–Lieb inequality, it seems likely that nonlinear versions will naturally arise in other areas of mathematics. Indeed, one of our primary motivations in [8] was to answer a question of Cowling, Martini, Müller and Parcet by showing that the sharp constant in Young's convolution inequality in a small neighbourhood of the identity in a general Lie group converges to the sharp constant (10) for the classical Young's convolution inequality as the neighbourhood shrinks.

In order to outline our proof of Theorem 3.3, we present an argument due to Keith Ball, for which the notation

$$\mathrm{BL}(\mathbf{L},\mathbf{c};\mathbf{f}) = \frac{\int_{\mathbb{R}^d} \prod_{j=1}^{m} (f_j \circ L_j)^{c_j}}{\prod_{j=1}^{m} (\int_{\mathbb{R}^{d_j}} f_j)^{c_j}}$$

is convenient. Since the integral of a convolution is the product of the integrals of the constituent functions, we get

$$\mathrm{BL}(\mathbf{L}, \mathbf{c}; \mathbf{f})\mathrm{BL}(\mathbf{L}, \mathbf{c}; \mathbf{g}) = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \prod_{j=1}^{m} [h_j^x(L_j(y))]^{c_j} \, \mathrm{d}y \right) \mathrm{d}x,$$

whenever $\int f_j = \int g_j = 1$ and where $h_j^x(z) = f_j(z)g_j(L_j(x) - z)$. If we write $\mathbf{h}^x = (h_j^x)_{j=1}^m$ and $\mathbf{f} * \mathbf{g} = (f_j * g_j)_{j=1}^m$, then we obtain

$$\mathrm{BL}(\mathbf{L}, \mathbf{c}; \mathbf{f})\mathrm{BL}(\mathbf{L}, \mathbf{c}; \mathbf{g}) \leq \sup_x \mathrm{BL}(\mathbf{L}, \mathbf{c}; \mathbf{h}^x)\mathrm{BL}(\mathbf{L}, \mathbf{c}; \mathbf{f} * \mathbf{g}).$$

This we refer to as Ball's inequality, and from it various bits of useful information may be extracted. Relevant to the proof of Theorem 3.3 is the fact that if we take $\mathbf{g}$ to be an *extremiser*, then we obtain

$$\mathrm{BL}(\mathbf{L}, \mathbf{c}; \mathbf{f}) \leq \sup_x \mathrm{BL}(\mathbf{L}, \mathbf{c}; \mathbf{h}^x). \tag{13}$$

If, in addition, we assume that $\mathbf{g}$ has *compact support*, then h_j^x becomes a localised version of f_j near $L_j(x)$, and therefore (13) contains the information that the functional $\mathbf{f} \mapsto \mathrm{BL}(\mathbf{L}, \mathbf{c}; \mathbf{f})$ increases under this localisation. This is the basis for the induction-on-scales argument that we employ, which is a reasonable strategy since, *if* we can carry out a similar argument with B_j rather than L_j, such a localisation process will effectively "linearise" the B_j. Implementing this strategy creates a number of technical obstacles (e.g. the above proof of Ball's inequality appears to heavily rely on the linearity of the L_j) and these will be outlined in the talk.

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