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COMBINATORICS OF MUTATIONS AND TORSION CLASSES

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Abstract. We consider the lattice $\mathsf{tors}\,A$ of torsion classes on a finite dimensional algebra. While this lattice is usually infinite, we show that it can still be well understood by studying its Hasse quiver. Moreover, we give some interpretation this Hasse quiver in terms of A-modules that permits to study *algebraic quotients of* $\mathsf{tors}\,A$, that is quotients of the form $\mathsf{tors}\,A \twoheadrightarrow \mathsf{tors}(A/I)$, $\mathscr{T} \mapsto \mathscr{T} \cap \operatorname{mod}(A/I)$ for an ideal I of A.

As the Hasse quiver of $\operatorname{tors} A$ contains naturally the exchange graph of support τ-tilting modules (as the subset consisting of functorially finite torsion classes), $\operatorname{tors} A$ can be viewed as a way to extend mutations, even though the behavior at non-functorially finite torsion classes changes drastically, as we will see in Subsection 2.4.

This document propose, in the first section, some reminders about torsion classes and fundamental results. The second section describes the lattice theory of torsion classes, mainly in the case where $\mathsf{tors}\,A$ is finite. The generalization to the case where $\mathsf{tors}\,A$ is infinite given in [DIR$^+$] will be sketched during the talk. The third part gives a quick overview of the relationship with $\mathbf{g}$-vectors.

1. Torsion classes

1.1. **Torsion pairs.** Here, we recall the notion of a *torsion pair* in a module category over a finite dimensional $\Bbbk$-algebra A over a field $\Bbbk$. It is a fundamental tool to study *tilting theory* and more generally derived equivalences. We start by giving a rough explanation about tilting theory and derived equivalences. Recall that the bounded derived category $\mathscr{D}^{\mathrm{b}}(\operatorname{mod} A)$ is a triangulated category containing in a canonical way $\operatorname{mod} A$ (it is called the *heart* of the canonical t-structure). A celebrated case of derived equivalence $\mathscr{D}^{\mathrm{b}}(\operatorname{mod} A) \cong \mathscr{D}^{\mathrm{b}}(\operatorname{mod} B)$ for another finite dimensional algebra B, is called *tilting*. This is the case when there is a *tilting module* $T \in \operatorname{mod} A$ such that $\mathsf{End}_A(T)^{\mathsf{op}} \cong B$ (we take the opposite algebra here as we consider right modules and endomorphisms act left). In such a case, there is a derived equivalence $\mathscr{D}^{\mathrm{b}}(\operatorname{mod} A) \cong \mathscr{D}^{\mathrm{b}}(\operatorname{mod} B)$, satisfying the following properties. The pair $(\mathsf{Fac}\,T, T^{\perp})$ is a *torsion pair* in $\operatorname{mod} A$ where

$$\mathsf{Fac}\,T := \{X \in \operatorname{mod} A \mid \exists T^n \twoheadrightarrow X\} \quad \text{and} \quad T^{\perp} := \{X \in \operatorname{mod} A \mid \mathsf{Hom}_A(T, X) = 0\}.$$

Also, there is a torsion pair $({}^{\perp}(\mathsf{D}\,T), \mathsf{Sub}\,\mathsf{D}\,T)$ in $\operatorname{mod} B$ where $\mathsf{D}\,T := \mathsf{Hom}_{\Bbbk}(T, \Bbbk)$,

$${}^{\perp}(\mathsf{D}\,T) := \{X \in \operatorname{mod} B \mid \mathsf{Hom}_B(X, \mathsf{D}\,T) = 0\} \text{ and } \mathsf{Sub}\,\mathsf{D}\,T := \{X \in \operatorname{mod} B \mid \exists X \hookrightarrow (\mathsf{D}\,T)^n\}.$$

Moreover, inside $\mathscr{D}^{\mathrm{b}}(\operatorname{mod} A)$, identified with $\mathscr{D}^{\mathrm{b}}(\operatorname{mod} B)$, we have $\mathsf{Fac}\,T = \mathsf{Sub}\,\mathsf{D}\,T$ and $T^{\perp} = ({}^{\perp}\,\mathsf{D}\,T)[1]$ where $[1]$ is the suspension functor. We illustrate this situation by the following example

Example 1.1. We consider the quiver $Q = 1 \to 2 \to 3$. In order to develop this example, we recall the *Auslander-Reiten quiver* of $\operatorname{mod} \mathbb{C}Q$:

$$\begin{tikzcd}
 & & (\mathbb{C} \to \mathbb{C} \to \mathbb{C}) \arrow[dr] & & \\
 & (0 \to \mathbb{C} \to \mathbb{C}) \arrow[ur] \arrow[dr] & & (\mathbb{C} \to \mathbb{C} \to 0) \arrow[ll, dashed] \arrow[dr] & \\
(0 \to 0 \to \mathbb{C}) \arrow[ur] & & (0 \to \mathbb{C} \to 0) \arrow[ll, dashed] \arrow[ur] & & (\mathbb{C} \to 0 \to 0) \arrow[ll, dashed]
\end{tikzcd}$$

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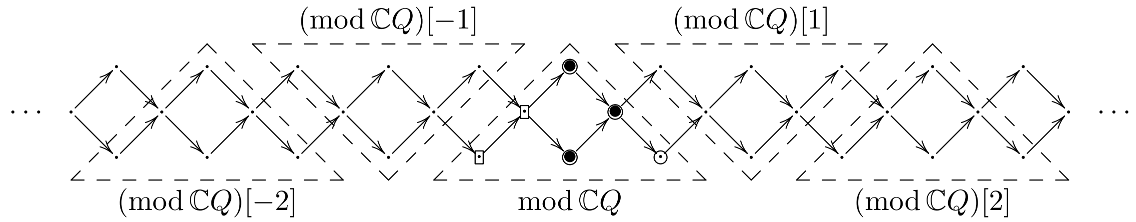
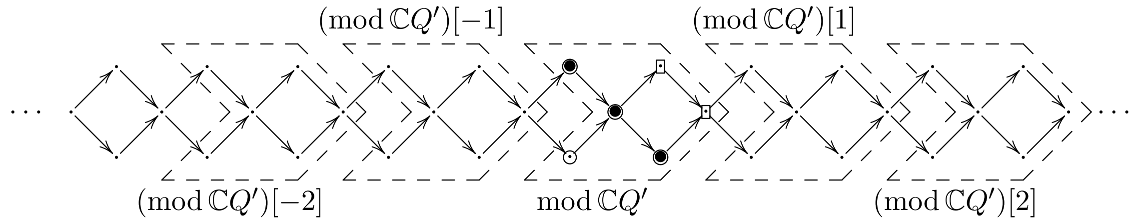
It turns out that, in this very simple case, called *hereditary*, the indecomposable objects of $\mathscr{D}^{\mathrm{b}}(\operatorname{mod}\mathbb{C}Q)$ are exactly shifted of representations of $\mathbb{C}Q$ and $\mathscr{D}^{\mathrm{b}}(\operatorname{mod}\mathbb{C}Q)$ admits an *Auslander-Reiten quiver*, which can be depicted as follows:



Moreover, the representation $T := (\mathbb{C} \to \mathbb{C} \to \mathbb{C}) \oplus (\mathbb{C} \to \mathbb{C} \to 0) \oplus (0 \to \mathbb{C} \to 0)$ depicted by thick dots in the picture is tilting and satisfies $\mathsf{End}_{\mathbb{C}Q}(T)^{\mathsf{op}} = \mathbb{C}Q'$ where $Q' = 1' \leftarrow 2' \to 3'$. Indecomposable modules of $\mathsf{Fac}\, T$ have been circled and indecomposable modules of $T^{\perp}$ has been framed. Then $\mathscr{D}^{\mathrm{b}}(\mathbb{C}Q)$ is equivalent with $\mathscr{D}^{\mathrm{b}}(\mathbb{C}Q')$ depicted here:



where $\mathsf{D}\, T = (\mathbb{C} \leftarrow 0 \to 0) \oplus (\mathbb{C} \leftarrow \mathbb{C} \to \mathbb{C}) \oplus (\mathbb{C} \leftarrow \mathbb{C} \to 0)$ has been thicken, $\mathsf{Sub}\,\mathsf{D}\, T = \mathsf{Fac}\, T$ has been circled and ${}^{\perp}(\mathsf{D}\, T) = (T^{\perp})[1]$ has been framed.

Tilting theory was first introduced by Brenner and Butler in [BB] and generalized in multiple fashions afterwards. See Subsection 1.2 for a generalization. As good classes of derived equivalences are produced by sequences of tilting, studying torsion pairs is a natural step toward understanding or classifying derived equivalences.

More precisely, a full subcategory $\mathscr{T}$ of $\operatorname{mod} A$ is a *torsion class* if it is closed under factor modules and extensions in $\operatorname{mod} A$. Dually, $\mathscr{F} \subseteq \operatorname{mod} A$ is a *torsion-free class* if it is closed under submodules and extensions in $\operatorname{mod} A$. We say that $(\mathscr{T}, \mathscr{F})$ is a *torsion pair* if $\mathscr{T}$ is a torsion class, $\mathscr{F}$ is a torsion-free class, and $\mathscr{F} = \mathscr{T}^{\perp}$ (or, equivalently, $\mathscr{T} = {}^{\perp}\mathscr{F}$). We denote the set of torsion (respectively, torsion-free) classes in $\operatorname{mod} A$ by $\mathsf{tors}\, A$ (respectively, $\mathsf{torsf}\, A$). An alternative characterization of torsion pairs is the following: $\mathscr{T}$ and $\mathscr{F}$ are full-subcategories of $\operatorname{mod} A$ such that for all $T \in \mathscr{T}$ and $F \in \mathscr{F}$, $\mathsf{Hom}_A(T, F) = 0$ and $\mathscr{T} * \mathscr{F} = \operatorname{mod} A$ where $\mathscr{T} * \mathscr{F}$ is the full subcategory of $\operatorname{mod} A$ consisting of X appearing as the middle term of a short exact sequence $0 \to T \to X \to F \to 0$ with $T \in \mathscr{T}$ and $F \in \mathscr{F}$.

We order $\mathsf{tors}\, A$ and $\mathsf{torsf}\, A$ by inclusion. Then there is an anti-isomorphisms of partially ordered set between them, namely $\mathscr{T} \mapsto \mathscr{T}^{\perp}$ and ${}^{\perp}\mathscr{F} \mapsfrom \mathscr{F}$. It turns out that $\mathsf{tors}\, A$ and $\mathsf{torsf}\, A$ are in fact complete lattices. In other terms, any family $(\mathscr{T}_i)_{i \in I}$ of torsion classes admits a *join* $\bigvee_{i \in I} \mathscr{T}_i$, that is a minimum torsion class among classes that are bigger than all $\mathscr{T}_i$'s and a *meet* $\bigwedge_{i \in I} \mathscr{T}_i$, that is a maximum torsion class among classes that are smaller than all $\mathscr{T}_i$'s and the same holds for torsion-free classes. In both cases, the meet is easy to construct: $\bigwedge_{i \in I} \mathscr{T}_i = \bigcap_{i \in I} \mathscr{T}_i$, and the join is obtained by using the anti-isomorphism above: $\bigvee_{i \in I} \mathscr{T}_i = {}^{\perp}\left(\bigcap_{i \in I} \mathscr{T}_i^{\perp}\right)$. Recall that the Hasse quiver $\mathsf{Hasse}\, L$ of an partially ordered set L has set of vertices L and an arrow $x \to y$ if and only if $x > y$ and there is no $z \in L$ with $x > z > y$.

Example 1.2. We continue with the quiver $1 \to 2 \to 3$ and the algebra $A := \mathbb{C}Q$. We give the Hasse quiver of $\mathsf{tors}\, A$ in Figure 1.3, using positions of modules in the Auslander-Reiten quiver depicted in Example 1.1.

1.2. **Torsion classes coming from τ-tilting modules.** A convenient way to study torsion classes consists of indexing them by certain modules. Unfortunately, torsion classes are not all of the form $\mathsf{Fac}\, T$ where T is a tilting module. Recall that, for a full subcategory $\mathscr{D}$ of $\operatorname{mod} A$ and

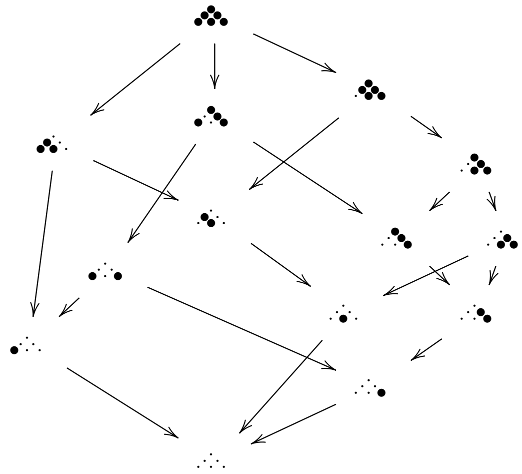


FIGURE 1.3. The lattice $\mathsf{tors}(\mathbb{C}Q)$

$X \in \operatorname{mod} A$, a *left $\mathscr{D}$-approximation of* X is a map $f : X \to D$ with $D \in \mathscr{D}$ such that any map from X to $D' \in \mathscr{D}$ factors through f. Dually, we define the notion of a *right $\mathscr{D}$-approximation of* X. We say that it is *functorially finite* if any A-module admits left and right $\mathscr{D}$-approximations. If $\mathscr{D}$ is a torsion class, the existence of a right-approximations is automatic, so the existence of left-approximations is enough.

We denote the set of functorially finite torsion (respectively, torsion-free) classes in $\operatorname{mod} A$ by $\mathsf{f}\text{-}\mathsf{tors}\, A$ (respectively, $\mathsf{f}\text{-}\mathsf{torsf}\, A$). We say that an A-module $M \in \mathscr{T}$ is Ext-*projective in* $\mathscr{T}$ if for all $N \in \mathscr{T}$ we have $\mathsf{Ext}^1_A(M, N) = 0$. The following result permits to start the investigation:

Proposition 1.4. [AS, Hos, Sma] *Let A be a finite dimensional algebra and $(\mathscr{T}, \mathscr{F})$ a torsion pair in $\operatorname{mod} A$. The following statements are equivalent:*
(a) *The torsion class $\mathscr{T}$ is functorially finite.*
(b) *The torsion-free class $\mathscr{F}$ is functorially finite.*
(c) *There exist a basic A-module $P(\mathscr{T}) \in \mathscr{T}$ such that $\mathsf{Fac}\, P(\mathscr{T}) = \mathscr{T}$ and $\mathsf{add}\, P(\mathscr{T})$ coincides with the class of Ext-projective A-modules in $\mathscr{T}$.*

If any of the above equivalent conditions hold, then the A-module $P(\mathscr{T})$ is a tilting $(A/\mathsf{ann}\,\mathscr{T})$-module.

Adachi–Iyama–Reiten [AIR] have started the study of modules $P(\mathscr{T})$ appearing in Proposition 1.4. They are called *support τ-tilting*. We give a brief introduction to τ-tilting theory.

An A-module M is *τ-rigid* if $\mathsf{Hom}_A(M, \tau M) = 0$ where τ is the Auslander-Reiten translation. We say that M is *τ-tilting* if, additionally, $|M| = |A|$ holds, where $|M|$ is the number of non-isomorphic indecomposable direct summands of M. Finally, we say that M is *support τ-tilting* if there exists an idempotent e of A such that M is a τ-tilting $(A/(e))$-module. We denote by $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ the set of isomorphism classes of basic support τ-tilting A-modules, by $\tau\text{-}\mathsf{rigid}\, A$ the set of isomorphism classes of τ-rigid A-modules, and by $\mathsf{i}\tau\text{-}\mathsf{rigid}\, A$ the set of isomorphism classes of indecomposable τ-rigid A-modules. By [AIR, §2.7], $M \mapsto \mathsf{Fac}\, M$ is a surjection from $\tau\text{-}\mathsf{rigid}\, A$ to $\mathsf{f}\text{-}\mathsf{tors}\, A$, which restricts to a bijection

$$\mathsf{Fac} : \mathsf{s}\tau\text{-}\mathsf{tilt}\, A \xrightarrow{\sim} \mathsf{f}\text{-}\mathsf{tors}\, A. \tag{1.5}$$

Moreover, this bijection is the inverse of $\mathscr{T} \mapsto P(\mathscr{T})$ in Proposition 1.4.

We also introduce the notion of a *τ-rigid pair.* A τ-rigid pair over A is a pair (M, P) where M is a τ-rigid A-module and P is a projective A-module satisfying $\mathsf{Hom}_A(P, M) = 0$. We say that (M, P) is basic if both M and P are. We denote by $\tau\text{-}\mathsf{rigid}\text{-}\mathsf{pair}\, A$ the set of isomorphism classes of τ-rigid pairs over A and by $\mathsf{i}\tau\text{-}\mathsf{rigid}\text{-}\mathsf{pair}\, A$ the subset of $\tau\text{-}\mathsf{rigid}\text{-}\mathsf{pair}\, A$ consisting of indecomposable ones (*i.e.* $(M, 0)$ with M indecomposable or $(0, P)$ with P indecomposable).

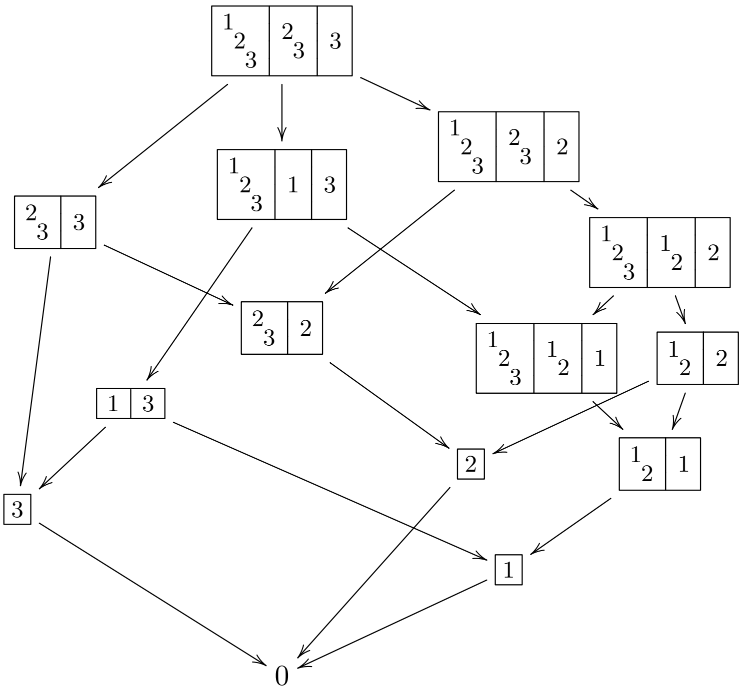


FIGURE 1.6. The lattice $\mathsf{s}\tau\text{-}\mathsf{tilt}(\mathbb{C}Q)$

We identify $M \in \tau\text{-}\mathsf{rigid}\, A$ with $(M, 0) \in \tau\text{-}\mathsf{rigid}\text{-}\mathsf{pair}\, A$. We say that a τ-rigid pair (M, P) is *τ-tilting* if, in addition, we have $|M| + |P| = |A|$. Notice that it is a maximality condition for the property to be τ-rigid. We denote by $\tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\, A$ the set of isomorphism classes of basic τ-tilting pair. We have a bijection $\tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\, A \to \mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ mapping (M, P) to M.

We lift the partial order on $\mathsf{f}\text{-}\mathsf{tors}\, A$ to a partial order on $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A \cong \tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\, A$ via Fac.

Example 1.7. We continue with our running example $Q = 1 \to 2 \to 3$. The Hasse quiver $\mathsf{Hasse}(\mathsf{s}\tau\text{-}\mathsf{tilt}\, \mathbb{C}Q)$ is drawn in Figure 1.6. It can be compared to Figure 1.3. We use here the composition series notation to describe indecomposable modules. Each digits represent a basis vector supported at the corresponding vertex of the quiver and non-zero matrix coefficients of the representation are going from top to bottom.

The order on $\tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\, A$ is characterized in the following way [AIR, Lemma 2.25]: For $(T, P), (U, Q) \in \tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\, A$, we have the inequality $(T, P) \geq (U, Q)$ if and only if $\mathsf{Hom}_A(U, \tau T) = 0$ and $\mathsf{Hom}_A(P, U) = 0$.

Moreover, $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A \cong \tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\, A$ is endowed with a mutation. We call a basic pair $(T, P) \in \tau\text{-}\mathsf{rigid}\text{-}\mathsf{pair}\, A$ *almost τ-tilting* if there exists $(X, Q) \in \mathsf{i}\tau\text{-}\mathsf{rigid}\text{-}\mathsf{pair}\, A$ such that $(T \oplus X, P \oplus Q)$ is τ-tilting.

Theorem 1.8. (a) [AIR, Theorem 2.18] *If (T, P) is an almost τ-tilting pair, there exist exactly two τ-tilting pairs (T_1, P_1) and (T_2, P_2) having (T, P) as a direct summand.*
(b) [AIR, Theorem 2.33] *The Hasse quiver of $\tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\, A$ has an arrow linking (T_1, P_1) and (T_2, P_2) of (a) and all arrows occur in this way.*

An easy consequence of Theorem 1.8 is that $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ (equivalently $\mathsf{f}\text{-}\mathsf{tors}\, A$) is Hasse-regular, *i.e.* all vertices of $\mathsf{Hasse}(\mathsf{s}\tau\text{-}\mathsf{tilt}\, A)$ have the same valency. In Theorem 1.8(b), the arrow goes from T_1 to T_2 if and only if $T_1 \notin \mathsf{Fac}\, T_2$. In this case, (T_1, P_1) is the biggest τ-tilting pair having (T, P) as a summand and (T_2, P_2) is the smallest one. We necessarily have $(T_1, P_1) = (T, P) \oplus (X, 0)$ for $X \in \mathsf{i}\tau\text{-}\mathsf{rigid}\, A$ and $(T_2, P_2) = (T, P) \oplus (X^*, Q)$ for $(X^*, Q) \in \mathsf{i}\tau\text{-}\mathsf{rigid}\text{-}\mathsf{pair}\, A$, and X^* is uniquely determined by the existence of an exact sequence $X \xrightarrow{u} \overline{T} \to X^* \to 0$, where u is a left minimal $\mathsf{add}(T)$-approximation of X.

Theorem 1.8 can be generalized to arbitrary $(X,Q) \in \tau\text{-}\mathsf{rigid\text{-}pair}\, A$ in the following way:

Theorem 1.9 ([AIR, Theorem 2.10])**.** *If $(X,Q) \in \tau\text{-}\mathsf{rigid\text{-}pair}\, A$ is basic, there is:*

- *A maximal $(T,P) \in \tau\text{-}\mathsf{tilt\text{-}pair}\, A$ having (X,Q) as a direct summand. It corresponds to the torsion class ${}^{\perp}(\tau X) \cap Q^{\perp}$ through the bijection Fac. We call it the* Bongartz completion of *(X,Q).*
- *A minimal $(T,P) \in \tau\text{-}\mathsf{tilt\text{-}pair}\, A$ having (X,Q) as a summand. It corresponds to the torsion class $\mathsf{Fac}\, X$ through the bijection Fac. We call it the* co-Bongartz completion of *X.*

2. Lattices of torsion classes and their quotients

If A is a finite dimensional $\Bbbk$-algebra and $B = A/I$ is a quotient of A by an ideal I, then there is a natural embedding $\operatorname{mod} B \subseteq \operatorname{mod} A$ which induces a surjection

$$\mathsf{tors}\, A \twoheadrightarrow \mathsf{tors}\, B, \mathscr{T} \mapsto \mathscr{T} \cap \operatorname{mod} B.$$

It turns out that this surjection is in fact a *lattice quotient* (*i.e.* it commutes with meet and join) [DIR$^+$]. The main aim of this section is to understand or even to characterize, in good cases, which lattice quotients of $\mathsf{tors}\, A$ can be realized as $\mathsf{tors}\, B$. Most results are proven in [DIR$^+$] with Iyama, Reading, Reiten and Thomas. A few results come from an earlier paper with Iyama and Jasso [DIJ].

A possible strategy to attack this problem consists in using τ-tilting modules. Unfortunately, as seen in Section 1.2, τ-tilting modules index functorially finite torsion classes and $\mathsf{f\text{-}tors}\, A$ is mostly never a lattice if it is not finite (see [IRTT] for more details). On the other hand, it is always a lattice when it is finite, as justified by the following result:

Theorem 2.1 ([DIJ, Theorem 1.2])**.** *The set $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ is finite if and only if every torsion class of $\operatorname{mod} A$ is functorially finite.*

Then, by Theorem 2.1, $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ is finite if and only if $\mathsf{f\text{-}tors}\, A = \mathsf{tors}\, A$ if and only if $\mathsf{tors}\, A$ is a finite lattice. We call an algebra A satisfying these conditions *τ-tilting finite*. We focus here on the case where $\mathsf{tors}\, A$ is finite, so that its lattice structure is intimately related with the mutation of τ-tilting pairs. A generalized version is proposed in [DIR$^+$], which is sketched in Subsection 2.4. The key point of Theorem 2.1 is the following one: We prove that for $T \in \mathsf{s}\tau\text{-}\mathsf{tilt}\, A$ and $\mathscr{T} \in \mathsf{tors}\, A$ such that $\mathsf{Fac}\, T \supsetneq \mathscr{T}$ (respectively $\mathscr{T} \supsetneq \mathsf{Fac}\, T$), there is an arrow $T \to U$ (respectively $U \to T$) in $\mathsf{Hasse}(\mathsf{s}\tau\text{-}\mathsf{tilt}\, A)$ such that $\mathsf{Fac}\, T \supsetneq \mathsf{Fac}\, U \supseteq \mathscr{T}$ (respectively $\mathscr{T} \supseteq \mathsf{Fac}\, U \supsetneq \mathsf{Fac}\, T$) and Theorem 2.1 follows by a combinatorial argument.

In order to study quotients $\mathsf{tors}\, A \twoheadrightarrow \mathsf{tors}\, B$, which we call *algebraic quotients*, we need first to understand general lattice quotients $\mathsf{tors}\, A \twoheadrightarrow L$. This is the object of the next subsection.

2.1. Lattice quotients of $\mathsf{tors}\, A$.

Let L be a finite lattice. We say that a lattice quotient $\pi : L \twoheadrightarrow L'$ *contracts* an arrow $q : x \to y$ of $\mathsf{Hasse}\, L$ if $\pi(x) = \pi(y)$. It is immediate that π is entirely determined by the set of arrows it contracts. However, the set of arrows contracted by a lattice quotient is subject to constrains. More precisely, we introduce the *forcing relation.* We say that an arrow q *forces* another arrow q' in $\mathsf{Hasse}\, L$, and we denote $q \rightsquigarrow q'$ if for any lattice quotient $\pi : L \twoheadrightarrow L'$, if π contracts q then π also contracts q'. This is clearly a preorder but not a partial order as it possesses non-trivial equivalence classes. We say that q and q' are *forcing-equivalent* if $q \rightsquigarrow q'$ and $q' \rightsquigarrow q$, and we denote $q \leftrightsquigarrow q'$.

Suppose now that A is τ-tilting finite. It turns out that $\mathsf{tors}\, A$ has a very nice structure called *polygonality*:

Proposition 2.2 ([DIR$^+$])**.** *The lattice $\mathsf{tors}\, A$ is* polygonal*. In other terms, for any two arrows $\mathscr{T} \to \mathscr{U}$ and $\mathscr{T} \to \mathscr{V}$ (respectively, $\mathscr{U} \to \mathscr{T}$ and $\mathscr{V} \to \mathscr{T}$) of $\mathsf{Hasse}(\mathsf{tors}\, A)$, the Hasse quiver of the segment $[\mathscr{U} \wedge \mathscr{V}, \mathscr{T}] := \{\mathscr{S} \mid \mathscr{U} \wedge \mathscr{V} \leq \mathscr{S} \leq \mathscr{T}\}$ (respectively, $[\mathscr{T}, \mathscr{U} \vee \mathscr{V}]$) is a polygon,* i.e. *it consists of its top element, its bottom element, and a disjoint union of exactly two chains.*

As a consequence, we have the following description of the forcing relation on $\mathsf{tors}\,A$: $\rightsquigarrow$ is the transitive closure of the relation $a \rightsquigarrow' b \rightsquigarrow' a$ and $a \rightsquigarrow' q_i$ for all polygons as follows:

$$\begin{tikzcd}
& \bullet \arrow[dl, "a"'] \arrow[dr, "b'"] & \\
\bullet \arrow[d, "q_1"'] & & \bullet \arrow[d, "q_{\ell+1}"] \\
\vdots \arrow[d, "q_\ell"'] & & \vdots \arrow[d, "q_{\ell+m}"] \\
\bullet \arrow[dr, "a'"'] & & \bullet \arrow[dl, "b"] \\
& \bullet &
\end{tikzcd}$$

2.2. **Brick labelling and categorification of the forcing relation.** One of the main tools to study lattice quotients of $\mathsf{tors}\,A$ is the following *categorification* of the forcing relation defined in Subsection 2.1. We first define the notion of the *brick labelling* of $\mathsf{Hasse}(\mathsf{s}\tau\text{-}\mathsf{tilt}\,A)$. A *brick* is an A-module the endomorphism algebra of which is a division algebra. A set $\{S_i\}_{i\in I}$ of bricks (or its direct sum) is called *semibrick* if $\mathsf{Hom}_A(S_i, S_j) = 0$ for any $i \neq j$. We denote by $\mathsf{brick}\,A$ the set of isomorphism classes of bricks of A, and by $\mathsf{sbrick}\,A$ the set of isomorphism classes of semibricks.

For a set $\mathscr{X}$ of A-modules, we denote by $\mathsf{T}(\mathscr{X})$ the smallest torsion class containing $\mathscr{X}$. We denote $\mathsf{f}\text{-}\mathsf{brick}\,A \subseteq \mathsf{brick}\,A$ the set of bricks S such that $\mathsf{T}(S)$ is functorially finite. A first result about bricks, proved with Iyama and Jasso is the following one:

Theorem 2.3 ([DIJ, Theorem 4.1])**.** *Let A be a finite dimensional algebra. Then there is a bijection*

$$\mathsf{i}\tau\text{-}\mathsf{rigid}\,A \to \mathsf{f}\text{-}\mathsf{brick}\,A$$

given by $X \mapsto X/\operatorname{rad}_E X$ for $E := \mathsf{End}_A(X)$.

As an application of Theorem 2.3, we give the following alternative characterization of τ-tilting finite algebras, which complements Theorem 2.1.

Theorem 2.4 ([DIJ, Theorem 4.2])**.** *Let A be a finite dimensional algebra. Then, the following conditions are equivalent.*

(i) *The algebra A is τ-tilting finite.*
(ii) *The set* $\mathsf{brick}\,A$ *is finite.*
(iii) *The set* $\mathsf{f}\text{-}\mathsf{brick}\,A$ *is finite.*

We now suppose that A is τ-tilting finite. By Theorem 2.1, all torsion classes of $\mathrm{mod}\,A$ are functorially finite, hence $\mathsf{f}\text{-}\mathsf{brick}\,A = \mathsf{brick}\,A$. We define a map

$$\mathsf{Hasse}_1(\mathsf{tors}\,A) = \mathsf{Hasse}_1(\mathsf{s}\tau\text{-}\mathsf{tilt}\,A) \to \mathsf{brick}\,A$$

called the *brick labelling.* Let Rad_A be the Jacobson radical of $\mathrm{mod}\,A$. Consider an arrow $q : T \to U$ of $\mathsf{Hasse}(\mathsf{s}\tau\text{-}\mathsf{tilt}\,A)$. Then there are decompositions $T = X \oplus M$ and $U = Y \oplus M$ for indecomposable A-modules X and Y and we prove that

$$S_q := \frac{X}{\mathsf{Rad}_A(T, X) \cdot T}$$

is a brick, as $\mathsf{Rad}_A(T, X) \cdot T \in \mathsf{Fac}\,T$, hence $\mathsf{Ext}^1_A(X, \mathsf{Rad}_A(T, X) \cdot T) = 0$.

Definition 2.5. We call S_q the *label* of the arrow q.

Example 2.6. Consider the algebra

$$\Lambda := \Bbbk\left(1 \xrightarrow{\alpha} 2 \underset{\beta^*}{\overset{\beta}{\rightleftarrows}} 3\right)\Big/(\alpha\beta, \beta\beta^*, \beta^*\beta).$$

We depict $\mathsf{Hasse}(\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda)$ in Figure 2.9. Labels of arrows are circled.

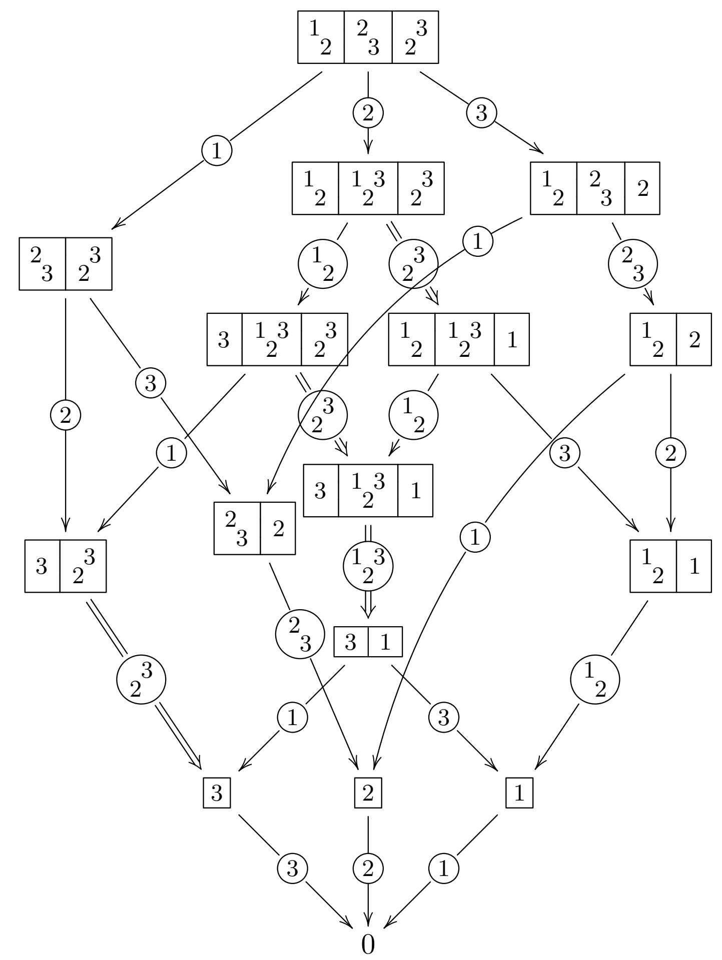


FIGURE 2.9. Hasse quiver of the lattice $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda$

It turns out that the brick labelling categorifies the forcing order in the following sense. For $\mathcal{S} \in \mathsf{sbrick}\,A$, denote by $\mathsf{Filt}\,\mathcal{S}$ the full subcategory of $\operatorname{mod} A$ consisting of objects filtered by elements of $\mathcal{S}$. It is a *wide subcategory* of $\operatorname{mod} A$, *i.e.* it is closed under kernels, cokernels and extensions.

Theorem 2.7 ([DIR$^+$])**.** *Let q and q' be two arrows of* $\mathsf{Hasse}(\mathsf{tors}\,A)$.

(a) We have $q \leftrightsquigarrow q'$ if and only if $S_q = S_{q'}$.

(b) The forcing relation is the transitive closure of the following:

$$q \rightsquigarrow q' \text{ if } \exists(\{S_q\} \sqcup \mathcal{S}) \in \mathsf{sbrick}\,A, S_{q'} \in \mathsf{Filt}(\{S_q\} \sqcup \mathcal{S}) \setminus \mathsf{Filt}\,\mathcal{S}.$$

Thanks to Theorem 2.7(a), we consider $\rightsquigarrow$ as a partial order on $\mathsf{brick}\,A$, also denoted by $\rightsquigarrow$.

Example 2.8. In Figure 2.9, the arrows labelled by bricks that are forced by ${}_{2}{}^{3}$ are doubled.

Using Theorem 2.7 with Theorem 2.3, we get the following important lattice theoretical result. Recall that $\mathscr{T} \in \mathsf{tors}\,A$ is *join-irreducible* if it is not 0 and it cannot be written non-trivially as the join of two other torsion classes. Equivalently, $\mathscr{T}$ is join-irreducible if there is a unique arrow pointing from $\mathscr{T}$ in $\mathsf{Hasse}(\mathsf{tors}\,A)$. Dually, we define the notion of *meet-irreducible* torsion classes.

Theorem 2.10 ([DIR$^+$])**.** *The lattice* $\mathsf{tors}\,A$ *is* congruence uniform. *In other terms, there are bijections:*

- *From join-irreducible torsion classes to forcing-equivalence classes of arrows of* $\mathsf{Hasse}(\mathsf{tors}\,A)$ *mapping $\mathscr{T}$ to the class of the unique arrow pointing from $\mathscr{T}$;*

- *From meet-irreducible torsion classes to forcing-equivalence classes of arrows of* $\mathsf{Hasse}(\mathsf{tors}\, A)$ *mapping* $\mathscr{T}$ *to the class of the unique arrow pointing toward* $\mathscr{T}$.

The argument for Theorem 2.10 is an easy consequence of the following facts (and their duals): First of all, join-irreducible torsion classes are exactly the ones of the form $\mathsf{Fac}\, X$ for $X \in \mathsf{i}\tau\text{-}\mathsf{rigid}\, A$. Secondly, the label of the unique arrow pointing from $\mathsf{Fac}\, X$ in this case is $X/\operatorname{rad}_E X$ for $E = \mathsf{End}_A(X)$. Then we conclude by Theorem 2.3 and Theorem 2.7. Notice that the surjectivity of both maps described in Theorem 2.10 is a general property of finite lattices, so the non-trivial point is the injectivity. It is important as in this case the forcing relation induces partial orders on the set of join-irreducible elements and on the set of meet-irreducible elements.

Another, more indirect, consequence of Theorem 2.10 is

Theorem 2.11 ([DIR+]). (a) *For all* $(N, Q) \in \tau\text{-}\mathsf{rigid}\text{-}\mathsf{pair}\, A$, *the subcategory*

$$\mathscr{W}(N, Q) := {}^{\perp}(\tau N) \cap Q^{\perp} \cap N^{\perp}$$

is a wide subcategory of $\operatorname{mod} A$.

(b) *There is a bijection from* $\tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\, A$ *to the set of wide subcategories of* A, *mapping a pair* (T, P) *to* $\mathscr{W}(T/Y, P)$ *where* Y *is the minimal summand of* T *satisfying* $\mathsf{Fac}\, Y = \mathsf{Fac}\, T$.

2.3. **Algebraic quotients of the lattice of torsion classes.** Recall that if A is a finite dimensional $\Bbbk$-algebra, a lattice quotient $\mathsf{tors}\, A \twoheadrightarrow L$ is *algebraic* if it is of the form $\mathsf{tors}\, A \twoheadrightarrow \mathsf{tors}\, B$ for a quotient $B = A/I$. This subsection is devoted to understand these algebraic quotients from the point of view of the brick labelling.

Let us fix a quotient $B = A/I$. A first observation about algebraic quotients is that their understanding at the level of support τ-tilting modules is well-behaved:

Proposition 2.12 ([DIR+]). (a) *If* $X \in \tau\text{-}\mathsf{rigid}\, A$ *then* $B \otimes_A X \in \tau\text{-}\mathsf{rigid}\, B$.

(b) *We have a commutative diagram*

$$\begin{tikzcd}
\tau\text{-}\mathsf{rigid}\, A \arrow[r, "\mathsf{Fac}\, -"] \arrow[d, "B\otimes_A -"'] & \mathsf{tors}\, A \arrow[d, "\mathscr{T}\mapsto\mathscr{T}\cap\operatorname{mod} B"] \\
\tau\text{-}\mathsf{rigid}\, B \arrow[r, "\mathsf{Fac}\, -"] & \mathsf{tors}\, B.
\end{tikzcd}$$

Notice that the vertical arrow $B \otimes_A -$ of Proposition 2.12 is not necessarily surjective in general. However, in the case where A is τ-tilting finite, horizontal arrows are isomorphisms of lattices and both vertical arrows are surjective.

The main result concerning algebraic quotient from the point of view of brick labelling is the following one:

Theorem 2.14 ([DIR+]). *Let* A *be a finite-dimensional* $\Bbbk$*-algebra that is* τ*-tilting finite, and* I *be an ideal of* A. *Then an arrow of* $\mathsf{Hasse}(\mathsf{tors}\, A)$ *is not contracted by* Θ_I *if and only if its label is in* $\operatorname{mod}(A/I)$. *Moreover, in this case, it has the same label in* $\mathsf{Hasse}(\mathsf{tors}\, A)$ *and* $\mathsf{Hasse}(\mathsf{tors}(A/I))$.

Example 2.15. We illustrate Theorem 2.14 by continuing Example 2.6. Let $\Lambda' := \Lambda/(\beta^*)$. Bricks that are not in $\operatorname{mod}(\Lambda')$ are the ones that have been doubled in Figure 2.9. We provide $\mathsf{Hasse}(\mathsf{s}\tau\text{-}\mathsf{tilt}(\Lambda'))$ in Figure 2.13, endowed with brick labelling to check Theorem 2.14. Notice that the same process can be applied to get Figure 2.13 from Figure 1.6, as Λ' is also a quotient of $\Bbbk(1 \to 2 \to 3)$.

We get the following corollary of Theorem 2.14.

Corollary 2.16 ([DIR+]). *Let* A *be a finite-dimensional* $\Bbbk$*-algebra that is* τ*-tilting finite and* I *be an ideal of* A. *Then the following are equivalent:*

(i) $I \subseteq I_0 := \bigcap_{S\in\mathsf{brick}\, A} \operatorname{ann} S$ *where* $\operatorname{ann} S := \{a \in A \mid aS = 0\}$;

(ii) *The map* $\mathscr{T} \mapsto \mathscr{T} \cap \operatorname{mod}(A/I)$ *is an isomorphism from* $\mathsf{tors}\, A$ *to* $\mathsf{tors}(A/I)$.

In particular, I_0 *is the maximum ideal of* A *satisfying each of these properties.*

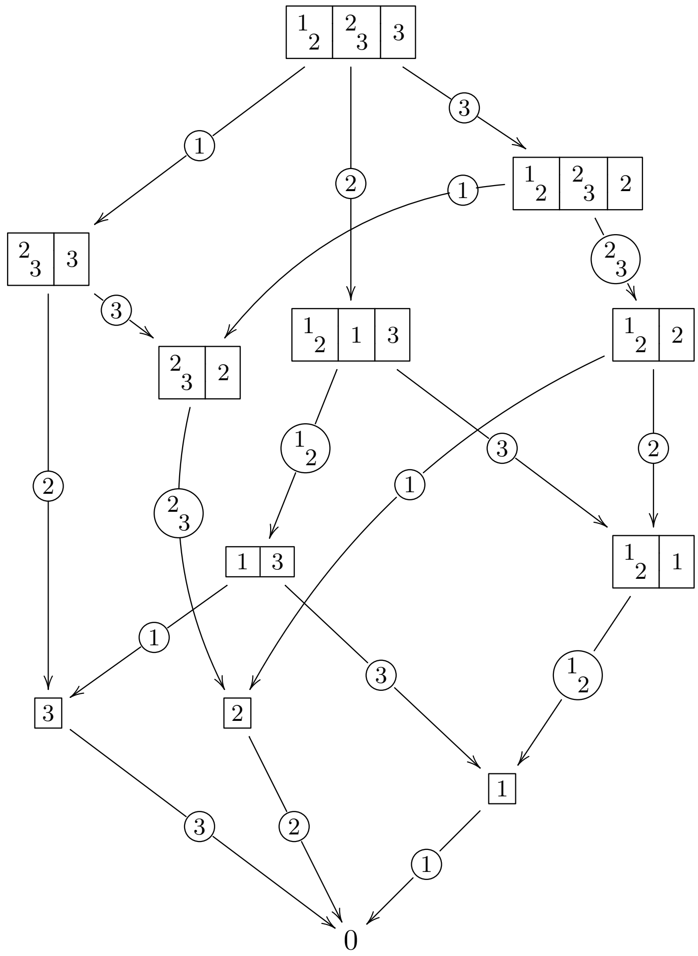


FIGURE 2.13. Hasse quiver of the lattice $\mathsf{s}\tau\text{-}\mathsf{tilt}\,\Lambda'$

It permits to recover easily the following result by [EJR] in the τ-tilting finite case.

Corollary 2.17. *Let A be a finite dimensional $\Bbbk$-algebra that is τ-tilting finite and Z the center of A. Then for any $I \subset A \operatorname{rad} Z$, $\eta_A(I)$ is the trivial congruence.*

We now give a lattice theoretical insight about the application that maps an ideal I to the lattice quotient $\mathsf{tors}\, A \twoheadrightarrow \mathsf{tors}(A/I)$. We denote by $\mathsf{Ideals}\, A$ the lattice of ideals of A. We denote by $\mathsf{Con}(\mathsf{tors}\, A)$ the lattice of congruence of $\mathsf{tors}\, A$, *i.e.* of equivalence relations respecting joins and meets (they are exactly kernels of lattice quotients). A congruence Θ is bigger than a congruence Ξ if all arrows contracted by Ξ are also contracted by Θ. We have the following general results:

Proposition 2.18 ([DIR+]). *Let A be a finite dimensional algebra that is τ-tilting finite. The map $\eta_A : \mathsf{Ideals}\, A \to \mathsf{Con}(\mathsf{tors}\, A)$ mapping $I \in \mathsf{Ideals}\, A$ to the kernel Θ_I of $\mathsf{tors}\, A \twoheadrightarrow \mathsf{tors}(A/I)$ is order preserving and commutes with joins (but not with meets in general).*

2.4. **The infinite case.** Here, we give an example when $\mathsf{tors}\, A$ is infinite, and we explain roughly on this example how the concepts defined above generalize to this case. The general version is in [DIR+].

Let k be an algebraically closed field and Q the Kronecker quiver

$$\begin{tikzcd} 2 \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & 1 \end{tikzcd}$$

and $A = kQ$. For $(\lambda, \mu) \in k^2 \setminus \{(0,0)\}$, we consider the following brick in $\operatorname{mod} A$:

$$S_{(\lambda:\mu)} = \left[\begin{tikzcd} 2 \arrow[d, bend right, "\lambda"'] \arrow[d, bend left, "\mu"] \\ 1 \end{tikzcd}\right]$$

whose isomorphism class only depends of $(\lambda : \mu) \in \mathbb{P}^1(k)$. Then, for $\mathscr{S} \subseteq \mathbb{P}^1(k)$ non-empty, we define the torsion class $\mathscr{T}(\mathscr{S}) = \mathsf{Filt}(\mathscr{S} \cup \{2\})$. We also define the torsion class $\mathscr{T}(\emptyset) = \bigcap_{\mathscr{S} \neq \emptyset} \mathscr{T}(\mathscr{S})$. Then $\mathscr{T} : 2^{\mathbb{P}^1(k)} \to \mathsf{tors}\, A$ is an injective morphism of complete lattices from the power set of $\mathbb{P}^1(k)$ to $\mathsf{tors}\, A$. We denote by $\mathcal{R}$ its image. Then, using classical knowledge about the Auslander-Reiten quiver of A, the labelled Hasse quiver of $\mathsf{tors}\, A$ is given by

$$\begin{tikzcd}
 & & \mathsf{add}\, S_1 \arrow[rrd, "1"] & & \\
\operatorname{mod} A \arrow[rru, "2"] \arrow[d, "1"] & & & & 0. \\
\mathsf{Fac}\, \begin{smallmatrix} & 2 & \\ 1 & & 1 \end{smallmatrix} \arrow[r, "\begin{smallmatrix} & 2 & \\ 1 & & 1 \end{smallmatrix}"] & \mathsf{Fac}\, \begin{smallmatrix} & 2 & & 2 & \\ 1 & & 1 & & 1 \end{smallmatrix} \arrow[r, dotted, no head] & \mathcal{R} \arrow[r, dotted, no head] & \mathsf{Fac}\, \begin{smallmatrix} 2 & & 2 \\ & 1 & \end{smallmatrix} \arrow[r, "\begin{smallmatrix} 2 & & 2 \\ & 1 & \end{smallmatrix}"] & \mathsf{add}\, S_2 \arrow[u, "2"]
\end{tikzcd}$$

Any arrow of $\mathsf{Hasse}\, \mathcal{R}$ has the form $q : \mathscr{T}(\mathscr{S}) \to \mathscr{T}(\mathscr{S}')$ for some $\mathscr{S}, \mathscr{S}' \subseteq \mathbb{P}^1(k)$ satisfying $\mathscr{S} \setminus \mathscr{S}' = \{(\lambda : \mu)\}$ for some $(\lambda : \mu) \in \mathbb{P}^1(k)$. The brick that labels this arrow is $S_q = S_{(\lambda:\mu)}$. To be more explicit, if P is an indecomposable preprojective module distinct from S_1, then $\mathsf{Fac}\, P$ contains all indecomposable modules except the ones that are to its left in the Auslander-Reiten quiver, if I is indecomposable preinjective, then $\mathsf{Fac}\, I$ contains I and indecomposable modules that are to its right in the Auslander-Reiten quiver. Finally, $\mathscr{T}(\mathscr{S})$ contains no preprojective modules, all preinjective modules and the tubes that are indexed by elements of $\mathscr{S}$.

This example is the starting point of a work in progress with Aaron Chan [CD] concerning idempotent quotients of Brauer graph algebras.

3. Lattices of torsion classes and Grothedieck groups

In this subsection, we investigate a question that is closely related to the lattice structure of torsion classes. Namely, we study the *fan* of $\mathbf{g}$-vectors of functorially finite torsion classes over an algebra, or equivalently, thanks to Proposition 1.4, of τ-rigid modules. As before, A is a finite dimensional algebra over a field $\Bbbk$. We fix a family of orthogonal primitive idempotents $e_1, e_2, \dots, e_n$ of A. Most of the discussion comes from [DIJ]. Notice that more progress have been made recently in [Asa, Yur] and that these results have a strong connection to cluster algebras and scattering diagrams [Bri, GHKK].

Consider a τ-rigid A-module X. As $\operatorname{mod} A$ has enough projective objects, there is an exact sequence $P_X^1 \xrightarrow{u} P_X^0 \xrightarrow{v} X \to 0$ with u and v right minimal and P_X^1 and P_X^0 projective. Then $P_X^\bullet := (P_X^1 \to P_X^0)$ is uniquely determined (up to non-unique isomorphism) and is called the *minimal projective presentation of* X. Additionally, $P_X^\bullet$ is *presilting*, *i.e.* $\mathsf{Hom}_{\mathscr{K}^{\mathrm{b}}(\mathsf{proj}\, A)}(P_X^\bullet, P_X^\bullet[n]) = 0$ for any $n > 0$ (we refer to classical textbooks, *e.g.* [Hap], for more details about the homotopy category $\mathscr{K}^{\mathrm{b}}(\mathsf{proj}\, A)$). More generally, if $(X, P) \in \tau\text{-}\mathsf{rigid\text{-}pair}\, A$, then its *minimal presentation*

$$P_{(X,P)} := P_X^\bullet \oplus P[1] = (P_X^1 \oplus P \to P_X^0)$$

is also presilting. If, additionally, (X, P) is a τ-tilting pair, then $P_X^\bullet \oplus P[1]$ is *silting*, *i.e.* maximal presilting up to multiplicities of direct summands.

We get in [DIJ, Proposition 6.3] that, for $(X, P) \in \tau\text{-}\mathsf{rigid\text{-}pair}\, A$, $P^0_{(X,P)}$ and $P^1_{(X,P)}$ have no common direct summands. Therefore, we define the $\mathbf{g}$*-vector* $\mathbf{g}_{(X,P)} = \mathbf{g} \in \mathbb{Z}^n$ in such a way that

$$P_{(X,P)} \cong \left(\bigoplus_{g_i < 0} (Ae_i)^{-g_i} \to \bigoplus_{g_i > 0} (Ae_i)^{g_i} \right).$$

Pushing the investigation further, we obtain that $(X, P) \in \tau\text{-}\mathsf{rigid\text{-}pair}\, A$ is uniquely determined by $\mathbf{g}_{(X,P)}$ up to isomorphism [DIJ, Proposition 6.5]. One of the reasons to investigate

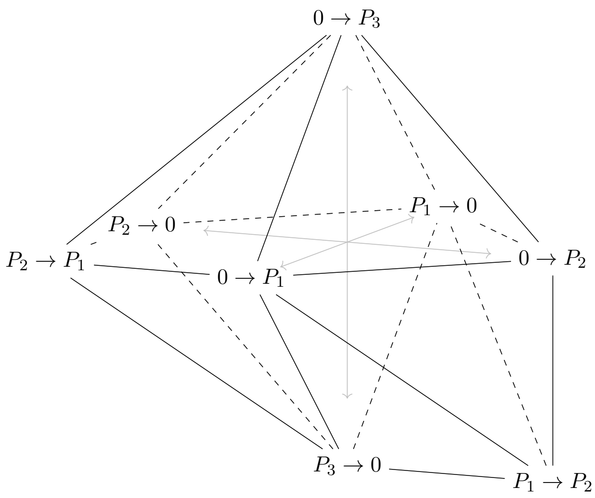


FIGURE 3.1. Geometric realization of $\Delta(\Lambda')$

g-vectors of τ-rigid modules is that they generalize the notion of **g**-vectors of cluster-tilting objects in cluster categories in a natural way (see [DIJ, §6.4]). In particular, they also satisfy important properties like sign-coherence. Our point of view here is that it also enrich in a certain way the partial order structure of $\mathsf{s}\tau\text{-}\mathsf{tilt}\, A$.

Consider $(T,P) \in \tau\text{-}\mathsf{rigid}\text{-}\mathsf{pair}\, A$. We denote $C(T,P)$ the cone generated by **g**-vectors of direct summands of (T,P) in $\mathbb{R}^n$. Then we get:

Theorem 3.2 ([DIJ, Theorem 6.6])**.** *Let $(T_1,P_1),(T_2,P_2) \in \tau\text{-}\mathsf{rigid}\text{-}\mathsf{pair}\, A$. Then $C(T_1,P_1) \cap C(T_2,P_2) = C(U,Q)$ where $(U,Q) \in \tau\text{-}\mathsf{rigid}\text{-}\mathsf{pair}\, A$ satisfies $\mathsf{add}(U,Q) = \mathsf{add}(T_1,P_1) \cap \mathsf{add}(T_2,P_2)$. In particular, $C(T_1,P_1)$ and $C(T_2,P_2)$ intersect at their boundaries if $\mathsf{add}(T_1,P_1) \neq \mathsf{add}(T_2,P_2)$.*

Theorem 3.2 permits to define a geometric simplicial complex $\Delta(A)$ attached to A. This complex is dual to $\mathsf{Hasse}(\tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\, A)$ (in the sense that vertices of $\mathsf{Hasse}(\tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\, A)$ correspond naturally to maximal n-cells of $\Delta(A)$ and arrows of $\mathsf{Hasse}(\tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\, A)$ to $(n-1)$ cells of $\Delta(A)$).

Example 3.3. As in Example 2.15, let Λ' be the algebra given by the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ subject to the relation $\alpha\beta = 0$. The complex $\Delta(\Lambda')$ is illustrated in Figure 3.1. We replaced indecomposable τ-tilting pairs by their presentations.

A natural question arising from the previous discussion is the following one:

Question 3.4. *Is the partial order on $\tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\, A$ entirely determined by the complex $\Delta(A)$?*

We give a partial answer to Question 3.4:

Theorem 3.5 ([DIJ, Theorem 6.11])**.** *Let $(X,P),(Y,Q) \in \tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\, A$ that are mutation of each other (*i.e.* neighbours in $\mathsf{Hasse}(\tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\, A)$). Then the following conditions are equivalent, where $(L,R) \in \tau\text{-}\mathsf{rigid}\text{-}\mathsf{pair}\, A$ is an object satisfying $\mathsf{add}(X,P) \cap \mathsf{add}(Y,Q) = \mathsf{add}(L,R)$.*

(i) $(X,P) > (Y,Q)$.
(ii) $\mathbb{R}^n_{\geq 0}$ *and $C(X,P)$ are contained in the same closed half-space defined by* $\mathsf{span}\, C(L,R)$.
(iii) $\mathbb{R}^n_{\leq 0}$ *and $C(Y,Q)$ are contained in the same closed half-space defined by* $\mathsf{span}\, C(L,R)$.

In other terms, $\Delta(A)$ determines entirely $\mathsf{Hasse}(\tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\, A)$. In particular, if A is τ-tilting finite, it determines the partial order on $\tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\, A$. We conjecture in [DIJ, Conjecture 6.14] that it is also true for τ-tilting infinite algebras.

We are also interested in the geometry of $\Delta(A)$. It is elementary to prove that if A is τ-tilting finite, then $\Delta(A)$ covers all $\mathbb{R}^n$ (see also [DIJ, §5]).

In [DIJ], we also prove that if A is τ-tilting finite, then $\Delta(A)$ has the so-called combinatorial property to be *shellable*. A consequence of this property is a strong analogous of simple connectedness. To give this property, let us start by a definition.

Definition 3.6. We say that a non-oriented cycle γ of $\mathsf{Hasse}(\tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\, A)$ has *rank at most* ℓ if there exists $(X, P) \in \tau\text{-}\mathsf{rigid}\text{-}\mathsf{pair}\, A$ with $|A| - \ell$ indecomposable summands such that all vertices γ passes through have (X, P) as a direct summand.

Let γ and δ be two non-oriented cycles of $\mathsf{Hasse}(\tau\text{-}\mathsf{tilt}\text{-}\mathsf{pair}\, A)$. We say that γ and δ are *related by cycles of rank* ℓ *in one step* if, up to cyclic rotation and change of orientation of γ and δ, we can write $\gamma = \varepsilon u$ and $\delta = \varepsilon v$ in such a way that uv^{-1} has rank at most ℓ. We say that γ and δ are *related by cycles of rank* ℓ if there exist a sequence $\gamma = \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k = \delta$ such that for any $i = 1, 2, \ldots, k$, ε_i is related to ε_{i-1} by cycles of rank ℓ in one step.

Then we get:

Theorem 3.7 ([DIJ, Theorem 5.5])**.** *Suppose that* A *is* τ*-tilting finite. Then any non-oriented cycle of* $\mathsf{Hasse}(\mathsf{tors}\, A)$ *is related by cycles of rank* 2 *to a trivial cycle.*

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