Asymptotic stability of the gradient flow of some nonlocal energies

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1. Introduction

In this short note we are going to collect a few results recently obtained in [1, 9, 10] concerning the H^{-1} gradient flow of an elastic energy of the type

$$\mathcal{J}(F) := \frac{1}{2}\int_{\Omega\setminus F} \mathbb{C}E(u_F) : E(u_F)\,dx + \mathcal{H}^2(\partial F).$$

Here the set $F \subset\subset \Omega$ represents the shape of a void within an elastic body $\Omega \subset\subset \mathbb{R}^3$ and u_F denotes the elastic equilibrium in $\Omega\setminus F$, i.e., the minimizer of the elastic energy

$$\mathcal{E}_F(u) := \frac{1}{2}\int_{\Omega\setminus F} \mathbb{C}E(u) : E(u)\,dx$$

among all displacements $u \in H^1(\Omega\setminus F;\mathbb{R}^3)$ satisfying the Dirichlet boundary condition $u = w_o$ on $\partial\Omega$. As usual, $E(u) := (Du + (Du)^T)/2$ stands for the symmetric part of the gradient of u while $\mathbb{C}$ is the elastic tensor acting on 3×3 matrices A and satisfying the ellipticity condition $\mathbb{C}A : A > 0$ for all $A \neq 0$. Finally, $\mathcal{H}^2$ is the two dimensional Hausdorff measure in $\mathbb{R}^3$.

The above energy is used to describe the equilibrium shapes of voids in elastically stresses solids, see for instance [15]. Existence and regularity of minimizers have been studied in two dimension in [8, 5], while a relaxation result for a related, similar functional in any dimension has been obtained in [3]. Note that if $F \subset\subset \Omega$ is a C^2 minimizer of the functional $\mathcal{J}$ under a volume constraint then F satisfies the following Euler-Lagrange equation

$$H_F - \frac{1}{2}\int_{\Omega\setminus F} \mathbb{C}E(u_F) : E(u_F) = \lambda \qquad \text{on } \partial F \tag{1.1}$$

for a suitable Lagrange multiplier $\lambda \in \mathbb{R}$, where H_F denotes the *mean curvature*, i.e., the sum of the principal curvatures of ∂F. In turn the elastic equilibrium u_F satisfies the following Dirichlet-Neumann problem

$$\begin{cases} \operatorname{div}\mathbb{C}E(u_F) = 0 & \text{in } \ \Omega\setminus F, \\ \mathbb{C}E(u_F)[\nu_F] = 0 & \text{on } \ \partial F, \\ u_F = w_o & \text{on } \ \partial\Omega. \end{cases} \tag{1.2}$$

Assume now that a chemical potential μ_t is acting on an initial configuration F_o. Then, according to the Einstein-Nerst law, the system starts to evolve according to the following equation

$$V_t = \Delta_{\partial F_t}\mu_t \qquad \text{on } \partial F_t,$$

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where V_t denotes the normal component of the velocity of the evolving surface F_t in the direction of the exterior normal ν_{F_t} and $\Delta_{\partial F_t}$ stands for the Laplace-Beltrami operator on ∂F_t. Assuming that μ_t is given by the first variation of the energy functional $\mathcal{J}$, i.e., by the left hand side of (1.1), the above equation becomes

$$V_t = \Delta_{\partial F_t}(H_{F_t} - W(E(u_{F_t}))), \tag{1.3}$$

where we have set $W(A) = \frac{1}{2}\mathbb{C}A : A$. Note that the above equation is *volume preserving* since a simple calculation shows that

$$\frac{d}{dt}|F_t| = \int_{\partial F_t} V_t \, d\mathcal{H}^2 = \int_{\partial F_t} \Delta_{\partial F_t}\big(H_{F_t} - W(E(u_{F_t}))\big) \, d\mathcal{H}^2 = 0.$$

Moreover, if $w_o = 0$ then $u_{F_t} = 0$ for all $t > 0$ and (1.3) reduces to the well known *surface diffusion* equation

$$V_t = \Delta_{\partial F_t} H_{F_t}. \tag{1.4}$$

It was first observed by Cahn and Taylor in [4] that the surface diffusion equation can be seen as the gradient flow of the energy functional $\mathcal{J}$ with respect to a suitable distance in H^{-1}. The same is still true for the more general equation (1.3). Note also that the surface diffusion *decreases the perimeter* since

$$\frac{d}{dt}\mathcal{H}^2(\partial F_t) = \int_{\partial F_t} H_{F_t} V_t \, d\mathcal{H}^2 = \int_{\partial F_t} H_{F_t} \Delta_{\partial F_t} H_{F_t} \, d\mathcal{H}^2 = -\int_{\partial F_t} |\nabla H_{F_t}|^2 \, d\mathcal{H}^2 \le 0,$$

where ∇ stands for the tangential gradient on ∂F_t. However, this property is not true in general for the solutions of (1.3).

The papers [1, 9, 10] deal with two different kind of results. The first one is a short time existence result for equation (1.3). The approach is based on a fixed point argument on the map $f \mapsto W(E(u_t^f))$, where u_t^f is the solution of the flow

$$V_t = \Delta_{\partial F_t}(H_{F_t} - f).$$

The implementation of this simple idea is technically quite involved. The main difficulty is that one would like to control the trace of the elastic energy $W(E(u_F))$ on ∂F by means of the mean curvature H_F. More precisely, given an integer $k \ge 1$, one would like to prove a linear estimate of the $H^{k-1/2}$ norm of the trace of $W(E(u_F))$ on ∂F in terms of the H^{k-1} norm of the mean curvature H_F. However, such an estimate is not true in general. Observe for instance that in 3D an L^2 bound on H_F does not imply that u_F is bounded in $H^2(\Omega \setminus F)$. Thus, equation (1.3) cannot be viewed as a lower order perturbation of the surface diffusion equation. To face this difficulty one has to prove some delicate new estimates for the solutions of the linear elasticity system (1.2), see Theorem 3.2. Of course, the existence proof simplifies a lot if no elasticity term is present as for the surface diffusion equation, see [7].

Local in time existence is the best we may hope for equation (1.3), since an example of Giga and Ito, see [12], shows that even for the surface diffusion equation in 2D pinching may occur in finite time.

Here, we shall also discuss a few results concerning the long time existence and asymptotic behavior of solutions of of both equations (1.3) and (1.4). Precisely, we show that if one starts close to a strictly stable stationary point for the energy functional J then the solution exists for all times and the map

$$t \mapsto \int_{\partial F_t} |\nabla(H_{F_t} - W(E(u_{F_t}))|^2 \, d\mathcal{H}^2$$

decays exponentially fast. In turn, this implies the exponential convergence of the flow F_t to a stationary set F_∞. The key point in the proof of the decay of the above integral is provided by the following energy identity. If (F_t) is a sufficiently smooth solution in $(0,T)$ of the evolution equation (1.3) then, setting

$$R_t = H_{F_t} - W(E(u_{F_t})),$$

on has

$$\begin{aligned} (1.5) \quad \frac{d}{dt}\left(\int_{\partial F_t} |\nabla R_t|^2 \, d\mathcal{H}^2\right) &= -2\partial^2 \mathcal{J}(F_t)[\Delta_{F_t} R_t] \\ &\quad - 2\int_{\partial F_t} B_{F_t}[\nabla R_t, \nabla R_t]\,(\Delta_{F_t} R_t)\, d\mathcal{H}^2 + \int_{\partial F_t} H_{F_t} |\nabla R_t|^2\,(\Delta_{F_t} R_t)\, d\mathcal{H}^2, \end{aligned}$$

where, for a function $\varphi \in H^1(\partial F_t)$ with zero average on ∂F_t, we have denoted with $\partial^2 \mathcal{J}(F_t)[\varphi]$ the second variation of the energy associated with the volume preserving flow of initial velocity $\varphi \nu_{F_t}$, see Theorem 3.3, and $B_{F_t}[\cdot,\cdot]$ is the second fundamental form of ∂F_t. Thus, the rough idea of the proof is to show that if F_t stays sufficiently close to a strictly stable stationary point G of J, then

$$\partial^2 \mathcal{J}(F_t)[\Delta_{F_t} R_t] \geq c_o \|\Delta_{F_t} R_t\|_{H^1(\partial F_t)}$$

for a fixed constant $c_0 > 0$, while the last two integrals in (1.5) can be estimated by

$$\sigma \|\nabla(\Delta_{F_t} R_t)\|_{L^2(\partial F_t)}$$

where the constant σ becomes smaller and smaller as the distance from the initial datum F_o to the strictly stable stationary set G goes to zero.

This argument is used also in the special case of the surface diffusion equation where the results presented here extend the ones on the exponential stability of n-dimensional spheres ([7, 16]). Other known stability results for the surface diffusion flow in the case of infinite cylinders were obtained in [13, 14] while the stability of a two-dimensional triple junction configuration (under Neumann conditions) was obtained in [11].

2. Stability of the surface diffusion flow

In this section we present the stability result proved in [10] for the surface diffusion flow in the *unit flat torus* $\mathbb{T}^3$. To this end, recall that $\mathbb{T}^3$ is defined as the quotient of $\mathbb{R}^3$ with respect to the equivalence relation $x \sim y \iff x - y \in \mathbb{Z}^3$. The functional spaces $H^k(\mathbb{T}^3)$, $k \in \mathbb{N}$, can be identified with the subspace of $H^k_{loc}(\mathbb{R}^3)$ of functions that are one-periodic with respect to all coordinate directions. Similarly, the space $C^{k,\alpha}(\mathbb{T}^3)$, $\alpha \in (0,1)$ may be identified with the space of one-periodic functions in $C^{k,\alpha}(\mathbb{R}^3)$. In the same way we shall say that a set $F \subset \mathbb{T}^3$ is of class H^k, $C^{k,\alpha}$ or smooth if its one-periodic extension to $\mathbb{R}^3$ is of class H^k, $C^{k,\alpha}$ or smooth, respectively.

As mentioned in the introduction the surface diffusion flow is the H^{-1} gradient flow of the perimeter. We recall that in the periodic setting, the (relative) perimeter of a set $F \subset \mathbb{T}^3$ is defined as

$$J(F) := P_{\mathbb{T}^3}(F) := \sup\left\{ \int_F \operatorname{div} \varphi \, dz \, : \, \varphi \in C^1(\mathbb{T}^3; \mathbb{R}^3) \, , \|\varphi\|_\infty \leq 1 \right\}.$$

We now recall the first and second variation formulas for the perimeter. To this aim we need to set some notation. Given a C^2 set $F \subset \mathbb{T}^3$ we denote by $F_1, \dots, F_m$ its connected components. Then, given a vector field $X \in C^2(\mathbb{T}^3; \mathbb{R}^3)$, let $(\Phi_t)_{t \in (-\delta, \delta)}$ be the associated flow, that is the solution of

$$\begin{cases} \dfrac{\partial \Phi_t}{\partial t} = X(\Phi_t), \\ \Phi_0 = Id. \end{cases} \tag{2.1}$$

We say that the flow $(\Phi_t)_{t \in (-\delta, \delta)}$ is *admissible for F* if

$$|\Phi_t(F_i)| = |F_i| \qquad \text{for } t \in (-\delta, \delta) \text{ and } i = 1, \dots, m.$$

Note that when the flow associated with X is admissible, then for $i = 1, \dots, m$ we have

$$\int_{\partial F_i} X \cdot \nu_F \, d\mathcal{H}^2 = 0.$$

Next theorem gives the first and second variation formulas for the perimeter in the flat torus (see [6, 2]).

Theorem 2.1. *Let $F \subset \mathbb{T}^3$ be a set of class C^2 and $X \in C^2(\mathbb{T}^3; \mathbb{R}^3)$ a vector field such that the associated flow $(\Phi_t)_{t \in (-\delta, \delta)}$ is admissible. Then*

$$\frac{d}{dt} J(\Phi_t(F))\Big|_{t=0} = \int_{\partial F} H_{\partial F} X \cdot \nu_F \, d\mathcal{H}^2 \, ,$$

where ν_F denotes the outer unit normal to ∂E. Moreover

$$\begin{aligned} \frac{d^2}{dt^2} J(\Phi_t(F))\Big|_{t=0} = & \int_{\partial F} \Big(|\nabla (X \cdot \nu_F)|^2 - |B_{\partial F}|^2 (X \cdot \nu_F)^2 \Big) \, d\mathcal{H}^2 \\ & - \int_{\partial F} H_{\partial F} \operatorname{div}_{\partial F} \big(X_\tau (X \cdot \nu_F) \big) \, d\mathcal{H}^2 + \int_{\partial F} H_{\partial F} (\operatorname{div} X)(X \cdot \nu_F) \, d\mathcal{H}^2 \, , \end{aligned}$$

where $|B_{\partial F}|^2$ is the sum of the squares of the principal curvatures of ∂F, $\operatorname{div}_{\partial F}$ stands for the tangential divergence and X_τ is the tangential component of X.

In view of the above formulas we say that a C^2 set $F \subset \mathbb{T}^3$ is *stationary* if its first variation with respect to any admissible flow is zero, i.e., there exist $\lambda_i \in \mathbb{R}$ such that

$$H_{\partial F_i} = \lambda_i \qquad \text{for all } i = 1, \dots, m.$$

Note that if F is stationary and the flow associated with the vector field X is admissible,

$$\frac{d^2}{dt^2} J(\Phi_t(F))\Big|_{t=0} = \int_{\partial F} \Big(|\nabla (X \cdot \nu_F)|^2 - |B_{\partial F}|^2 (X \cdot \nu_F)^2 \Big) \, d\mathcal{H}^2.$$

In view of this formula we to introduce the following quadratic form, defined on the functions $\varphi \in \mathcal{H}(\partial F) := \{\varphi \in H^1(\partial F) : \int_{\partial F_i} \varphi \, d\mathcal{H}^2 = 0, \text{ for all } i = 1, \dots, m\}$ as

$$\partial^2 J(F)[\varphi] := \int_{\partial F} \Big(|\nabla \varphi|^2 - |B_{\partial F}|^2 \varphi^2 \Big) \, d\mathcal{H}^2$$

Note that the condition $\int_{\partial F_i} \varphi \, d\mathcal{H}^2 = 0$ for $i = 1, \dots, m$ is related to the fact that we allow only variations preserving the volume of the connected components of F.

The notion of stability amounts to requiring that $\partial^2 J$ is positive definite in a suitable sense. In fact we have to take into account that J is translation invariant, so that in particular $J(F) = J(\cup_{i=1}^m (F_i + t\eta_i))$, where $\eta_i \in \mathbb{R}^3$, for $i = 1, \dots, m$ and $t \in \mathbb{R}$. By differentiating twice this identity with respect to t, one obtains that on the space

$$T(\partial F) := \{\varphi \in \mathcal{H}(\partial F) \, : \, \varphi = \eta_i \cdot \nu_F \text{ on } \partial F_i, \; \eta_i \in \mathbb{R}^3, i = 1, \dots, m\}$$

the second variation vanishes. In view of these observations, we say that the stationary set F is *strictly stable* if

$$\partial^2 J(F)[\varphi] > 0 \qquad \text{for all } \varphi \in T^\perp(\partial F) \setminus \{0\}.$$

With all these definition in hands we can now state the following stability result which is a particular case of the stability result of [10], see also [1]. To this end, we say that a family of sets F_t converges exponentially fast to a smooth set E if there exists t_o with the property that for all $t > t_0$

$$\partial F_t = \{x + h(x,t)\nu_E(t) : \, x \in \partial E\}$$

and for all integers $k \geq 1$ there exist $c_k > 0$ and $C_k > 1$ such that

$$\|h(\cdot, t)\|_{C^k(\partial F_\infty)} \leq C_k e^{-c_k t}.$$

Theorem 2.2. *Let $G \subset \mathbb{T}^3$ be a strictly stable smooth stationary set. There exists $\delta_o > 0$ with the following property. If F_o is such that $\partial F_o = \{x + h_o(x)\nu_G(x) : \, x \in \partial G\}$, where $\|h_o\|_{H^3(\partial G)} \leq \delta_o$ and $|F_o| = |G|$, the unique classical solution $(F_t)_{t>0}$ of the flow* (1.4) *with initial datum F_o exists for all times $t > 0$.*

Moreover $F_t \to \sigma + F_\infty$ exponentially fast, where F_∞ is the unique stationary set near G such that $|F_{\infty,i}| = |F_{o,i}|$ for $i = 1, \dots, m$. In particular, if $|F_{o,i}| = |G_i|$ for $i = 1, \dots, m$, then $F_t \to \sigma + G$ and $|\sigma|$ vanishes as $\delta_o \to 0^+$.

3. The surface diffusion flow with elasticity

In this section we discuss local in time existence and stability for equation (1.3).

To this end, we denote by $G \subset\subset \Omega$ a fixed reference set and choose $\eta > 0$ so that the signed distance function from ∂G, denoted by $\mathrm{d}(\cdot, \partial G)$, is smooth in the tubular neighborhood $\{x : \, |\mathrm{d}(x, \partial G)| < \eta\}$. Given $h \in C^1(\partial G)$ with $\|h\|_{L^\infty(\partial G)} < \eta$, we set

$$\partial F_h = \{x + h(x)\nu_G(x) : \, x \in \partial G\}. \tag{3.1}$$

Next local in time existence theorem is proved in [10]. To simply the notation, in the statement F_o stands for F_{h_o} while F_t stands for $F_{h(\cdot,t)}$.

Theorem 3.1. *Let $M>0$ be such that $||W(E(u_G))||_{L^\infty(\partial G)}<M/4$. There exist $T,\delta>0$ such that, if $\|h_o\|_{H^3(\partial G)}\le M$ and $\|h_o\|_{L^2(\partial G)}<\delta$, there is a unique solution $(F_t)_{t\in(0,T)}$ to (1.3) with initial datum F_o, with $h\in H^1(0,T;H^1(\partial G))\cap L^\infty(0,T;H^3(\partial G))$. Moreover, for every integer $k\ge 1$,*

$$\sup_{0\le t\le T} t^k\|h(\cdot,t)\|^2_{H^{2k+3}(\partial G)}+\int_0^T t^k\|h(\cdot,t)\|^2_{H^{2k+5}(\partial G)}\,dt\le C(k,M).$$

The proof of the above theorem requires precise estimates of solutions of the linear elasticity system (1.2). They are stated in the next theorem. To this end, if F_h is as in (3.1) we denote by $\lambda_h:\partial G\mapsto\partial F_h$ the map $\lambda_h(x)=x+h(x)\nu_G(x)$.

Theorem 3.2. . *Let $K>0$, $\alpha\in(0,1)$, and let $k\ge 3$ be an integer. There exists $C_k=C_k(K)>0$ such that if $h\in H^k(\partial G)$, $\|h\|_{C^{1,\alpha}}\le K$ and $\|h\|_{L^\infty}(\partial G)<\eta$ then*

$$\|W(E(u_{F_h}))\circ\lambda_h\|_{H^{k-\frac{3}{2}}(\partial G)}\le C_k(\|h\|_{H^k(\partial G)}+1).$$

Moreover there exists $C=C(K)>0$ such that, if h_1, $h_2\in H^3(\partial G)$ with $\|h_i\|_{H^3(\partial G)}\le K$, $\|h_i\|_{L^\infty(\partial G)}<\eta$ for $i=1,2$, then

$$\|u_{F_{h_2}}\circ\lambda_{h_2}-u_{F_{h_1}}\circ\lambda_{h_1}\|_{H^{3/2}(\partial G)}\le C\|h_2-h_1\|_{H^2(\partial G)}.$$

Let $\Gamma_{G,1}$, …, $\Gamma_{G,m}$ be the connected components of ∂G and let $G_1,\dots,G_m$ be the bounded open sets enclosed by them. Note that the G_i's are not in general the connected components of G as it may happen that $G_i\subset G_j$ for some $i\neq j$. Note also that if $F=F_h$ is as in (3.1) then ∂F has the same number m of connected components $\Gamma_{F,1}$, …, $\Gamma_{F,m}$, which can be numbered in such a way that

$$\Gamma_{F,i}=\{x+h(x)\nu_G(x):\,x\in\Gamma_{G,i}\}.$$

Similarly to what we have done in the previous section, we say that F is *stationary* if there exist $\lambda_i\in\mathbb{R}$ such that

$$H_{\partial F}-W(E(u_F))=\lambda_i\qquad\text{for all } i=1,\dots,m.$$

Given a vector field $X\in C^2_c(\Omega;\mathbb{R}^3)$, we say that the associated flow $(\Phi_t)_{t\in(-\delta,\delta)}$, i.e., the solution of (2.1), is *admissible for F* if

$$|\Phi_t(F_i)|=|F_i|\qquad\text{for } t\in(-\delta,\delta)\text{ and } i=1,\dots,m.$$

As before we may define the second variation corresponding to the vector field X as $\frac{d^2}{dt^2}\mathcal{J}(\Phi_t(F))\Big|_{t=0}$. Then we have

Theorem 3.3. *Let $F\subset\subset\Omega$ be a smooth set, $X\in C^2_c(\Omega;\mathbb{R}^3)$. Set $\psi:=X\cdot\nu_F$. Then,*

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{J}(\Phi_t(F))\Big|_{t=0}&=\int_{\partial F}|\nabla\psi|^2-|B_F|^2\psi^2\,d\mathcal{H}^2-2\int_{\Omega\setminus F}W(E(u_\psi))\,dx\\ &-\int_{\partial F}\partial_{\nu_F}(W(E(u_F)))\psi^2\,d\mathcal{H}^2-\int_{\partial F}(H_F-W(E(u_F)))\operatorname{div}_{\partial F}(\psi X_\tau)\,d\mathcal{H}^2,\end{aligned}\tag{3.2}$$

where the function u_ψ is the unique solution in $H^1(\Omega\setminus F;\mathbb{R}^3)$, with $u_\psi=0$ on $\partial\Omega$, of

$$\int_{\Omega\setminus F}\mathbb{C}E(u_\psi):E(\varphi)\,dx=-\int_{\partial F}\operatorname{div}_{\partial F}(\psi\,\mathbb{C}E(u_F))\cdot\varphi\,d\mathcal{H}^2$$

for all $\varphi\in H^1(\Omega\setminus F;\mathbb{R}^2)$ such that $\varphi=0$ on $\partial\Omega$.

Note that if F is stationary then the last integral in (3.2) is zero. Therefore, as before, we may define for any $\psi \in \mathcal{H}(\partial F)$

$$\partial^2 \mathcal{J}(F)[\psi] := \int_{\partial F} |\nabla \psi|^2 - |B_F|^2 \psi^2 \, d\mathcal{H}^2 \\ - 2 \int_{\Omega \setminus F} W(E(u_\psi)) \, dx - \int_{\partial F} \partial_{\nu_F}(W(E(u_F)))\psi^2 \, d\mathcal{H}^2.$$

Again, we say that a stationary set F is *strictly stable* if

$$\partial^2 J(F)[\varphi] > 0 \qquad \text{for all } \varphi \in \mathcal{H}(\partial F) \setminus \{0\}.$$

With these definitions in hand we may now state the following asymptotic stability result.

Theorem 3.4. *Let $G \subset\subset \Omega$ be a smooth strictly stable stationary set. There exists $\delta > 0$ such that if $h_o \in H^3(\partial G)$ with $\|h_o\|_{H^3(\partial G)} < \delta$, then the unique solution $(F_t)_{t>0}$ of the flow* (1.3) *with initial datum F_o is defined for all times $t > 0$.*

Moreover $F_t \to F_\infty$ exponentially fast, where F_∞ is the unique stationary set near G such that $|F_{\infty,i}| = |F_{o,i}|$ for $i = 1, \dots, m$. In particular, if $|F_{o,i}| = |G_i|$ for $i = 1, \dots, m$, then $F_t \to G$ exponentially fast.

References

[1] E. Acerbi, N. Fusco, V. Julin & M. Morini, *Nonlinear stability results for the modified Mullins-Sekerka and the surface diffusion flow.* J. Differential Geom., to appear.

[2] E. Acerbi, N. Fusco & M. Morini, *Minimality via second variation for a nonlocal isoperimetric problem.* Comm. Math. Phys. **322** (2013), 515–557.

[3] A. Braides, A. Chambolle & M.Solci, *A relaxation result for energies defined on pairs set-function and applications.* ESAIM Cont. Optim. Calc. Var. **13** (2007), 717–734.

[4] J.W. Cahn & J.E. Taylor, *Overview No.113 - Surface motion by surface-diffusion.* Acta Metallurgica et Materialia, **42** (1994), 1045–1063.

[5] G.M. Capriani, V. Julin, G. Pisante, *A quantitative second order minimality criterion for cavities in elastic bodies.* SIAM J. Math. Anal. **45** (2013), 1952–1991.

[6] R. Choksi & P. Sternberg, *On the first and second variations of a nonlocal isoperimetric problem,* J. Reine Angew. Math. **611** (2007), 75–108.

[7] J. Escher, U.F. Mayer, G. Simonett, *The surface diffusion flow for immersed hypersurfaces.* SIAM J. Math. Anal. **29** (1998), 1419–1433.

[8] I. Fonseca, N. Fusco, G. Leoni & V. Millot, *Material voids in elastic solids with anisotropic surface energies.* J. Math. Pures Appl. **96** (2011), 591–639.

[9] N. Fusco, V. Julin & M. Morini, *The surface diffusion flow with elasticity in the plane.* Comm. Math. Phys., **362** (2018), 571–607.

[10] N. Fusco, V. Julin & M. Morini, *The surface diffusion flow with elasticity in three dimensions.* Preprint (2018).

[11] H. Garcke, K. Ito & Y. Kohsaka, *Nonlinear stability of stationary solutions for surface diffusion with boundary conditions,* SIAM J. Math. Anal. **40** (2008), 491–515.

[12] Y. Giga & K. Ito, *On pinching of curves moved by surface diffusion.* Commun. Appl. Anal. **2** (1998), 393–405.

[13] J. LeCrone & G. Simonett, *On well-posedness, stability, and bifurcation for the axisymmetric surface diffusion flow.* SIAM J. Math. Anal. **45** (2013), 2834–2869.

[14] J. LeCrone & G. Simonett, *On the flow of non-axisymmetric perturbations of cylinders via surface diffusion.* J. Differential Equations **260** (2016), 5510–5531.

[15] A. Rätz, A. Ribalta & A. Voigt, *Surface evolution of elastically stressed films under deposition by a diffuse interface model.* J. Comp. Phys. **214** (2006), 187–208.

[16] G. Wheeler, *Surface diffusion flow near spheres.* Calc. Var. Partial Differential Equations **44** (2012), 131–151.