Gaussian beta ensembles in global regime*

Khanh Duy Trinh
Research Alliance Center for Mathematical Sciences, Tohoku University

Abstract

As a generalization of Gaussian orthogonal/unitary/symplectic ensembles, Gaussian beta ensembles, one of the most studied models in random matrix theory, were originally defined in terms of the joint density of eigenvalues. They have been studied by using some methods in statistical mechanics because the distributions of eigenvalues can be viewed as the equilibrium measures of a one-dimensional Coulomb log-gas with an external Gaussian potential. Gaussian beta ensembles are now realized as eigenvalues of certain random tridiagonal matrices. Since the discovery of the random matrix models, many new spectral properties of Gaussian beta ensembles have been established. This talk gives a brief survey on recent developments with emphasizing on the global regime which deals with the convergence to a limiting measure, and the fluctuation around the limit of the empirical distributions.

1 Introduction

Gaussian orthogonal (resp. unitary) ensembles (GOE and GUE for short) were introduced to model the nuclei of heavy atoms in the 1950s by the physicist E. Wigner. On the one hand, they are ensembles of real symmetric (resp. complex Hermite) random matrices with Gaussian entries. On the other hand, their distributions are invariant under orthogonal (resp. unitary) conjugations, and hence the name. They are among a few models in which the joint distribution of eigenvalues can be calculated explicitly. For GOE and GUE ($\beta = 1$ and $\beta = 2$, respectively in the following), all the eigenvalues $\{\lambda_i\}_{i=1}^N$ are real and their joint density is given by

$$\begin{aligned}
&\frac{1}{Z_{N,\beta}} \prod_{i<j} |\lambda_j - \lambda_i|^\beta \exp\Big(-\frac{N\beta}{4} \sum_{i=1}^N \lambda_i^2 \Big) \\
&= \frac{1}{Z_{N,\beta}} \exp\Big\{ \frac{\beta}{2} \Big(\sum_{i\neq j} \log|\lambda_j - \lambda_i| - N \sum_{i=1}^N V(\lambda_i) \Big) \Big\}, \qquad (1)
\end{aligned}$$

$Z_{N,\beta}$ being a normalizing constant, and $V(x) = x^2/2$. Gaussian symplectic ensembles (GSE) are another model with quaternion entries ($\beta = 4$). Note that the above joint density is still meaningful for all positive values of β, leading to define the so-called Gaussian beta ensembles (GβE). GβE can also be viewed as the equilibrium measure of a one dimensional Coulomb log-gas under the potential V at the inverse temperature β.

To a random matrix model (with real eigenvalues), we are interested in studying the limiting behavior of eigenvalues through three main regimes: global, local and edge regimes concerning with the limiting behavior of the empirical distribution, $L_N = N^{-1} \sum_{i=1}^N \delta_{\lambda_i}$, the local/bulk statistics at a reference energy E, $\xi_N(E) = \sum_{i=1}^N \delta_{N(\lambda_i - E)}$, and the largest eigenvalue $\lambda_{\max} = \max\{\lambda_i : 1 \le i \le N\}$, respectively. Here δ_λ denotes the Dirac measure.

Dumitriu and Edelman (2002) [5] introduced a random matrix model for GβE. They are symmetric tridiagonal matrices, called Jacobi matrices, with independent entries having certain distributions. To be more precise, the eigenvalues of the following matrix $T_{N,\beta}$ are distributed as

*This work is supported by JSPS KAKENHI Grant Numbers JP16K17616.

Gaussian beta ensembles

$$T_{N,\beta} = \frac{1}{\sqrt{N\beta}} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{(N-1)\beta} & & \\ \chi_{(N-1)\beta} & \mathcal{N}(0,2) & \chi_{(N-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \chi_\beta & \mathcal{N}(0,2) \end{pmatrix}. \tag{2}$$

Here $\mathcal{N}(0,2)$ denotes the Gaussian distribution with mean zero and variance 2 and χ_k denotes the chi distribution with k degrees of freedom. Since then, various features of the spectrum can be read off the tridiagonal matrices. Some remarkable results by Virág and his colleagues are: the convergence to β-Tracy-Widom distributions of the largest eigenvalue [9], and the convergence to Sine β point processes of the bulk statistics [14].

The talk focuses on the global regime of GβE. For fixed β, it is well known that the empirical distribution $L_{N,\beta} = N^{-1} \sum_{i=1}^N \delta_{\lambda_i}$, converges weakly to the semicircle distribution, almost surely (a.s.). This means that for any bounded continuous function $f\colon \mathbb{R} \to \mathbb{R}$,

$$\int f(x) dL_{N,\beta}(x) = \frac{1}{N} \sum_{i=1}^N f(\lambda_i) \to \int_{-2}^2 f(x) \frac{1}{2\pi} \sqrt{4-x^2} dx \text{ a.s. as } N \to \infty. \tag{3}$$

The Gaussian fluctuation around the limit was also well established: for smooth enough test function f,

$$\sqrt{\beta} \left(\sum_{i=1}^N f(\lambda_i) - \mathbb{E}\Big[\sum_{i=1}^N f(\lambda_i) \Big] \right) \xrightarrow{d} \mathcal{N}(0, \sigma_f^2), \tag{4}$$

where σ_f^2 does not depend on β and is given by

$$\sigma_f^2 = \frac{1}{2\pi^2} \int_{-2}^2 \int_{-2}^2 \left(\frac{f(x)-f(y)}{x-y} \right)^2 \frac{4-xy}{\sqrt{4-x^2}\sqrt{4-y^2}} dxdy. \tag{5}$$

Here '$\xrightarrow{d}$' denotes the convergence in distribution.

How do the above two convergence results depend on β? This talk gives the answer to that question. In particular, the above two convergences hold as long as $N\beta \to \infty$. In addition, the class of test functions for which the above Gaussian fluctuation holds is now known to contain all functions f having continuous derivative of polynomial growth.

This talk also introduces several results in case where $N\beta \to 2c \in (0, \infty)$. In this case, almost surely, the empirical distribution $L_{N,\beta}$ converges weakly to a probability measure μ_c with density

$$\mu_c(x) = \sqrt{c} \rho_c(\sqrt{c} x), \tag{6}$$

$$\rho_c(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \frac{1}{|\hat{f}_c(x)|^2}, \text{where } \hat{f}_c(x) = \sqrt{\frac{c}{\Gamma(c)}} \int_0^\infty t^{c-1} e^{-\frac{t^2}{2} + ixt} dt. \tag{7}$$

Note that the measure ρ_c, studied by Askey and Wimp (1984) [2], is called the probability measure of associated Hermite polynomials. It is worth mentioning that in the local regime, the local/bulk statistics $\xi_{N,\beta}(E) = \sum_{i=1}^N \delta_{N(\lambda_i - E)}$ converges weakly to a homogeneous Poisson point process with intensity $\mu_c(E)$ (see Benaych-Georges and Péché (2015) [3] and Nakano and Trinh (2018) [8]). The edge regime is open.

2 Wigner's semicircle law regime

This section introduces several approaches to show Wigner's semicircle law.

Theorem 2.1 (Wigner's semicircle law). *As $N \to \infty$ with $N\beta \to \infty$, almost surely, the empirical distribution $L_{N,\beta}$ converges weakly to the semicircle distribution, meaning that for any bounded continuous function $f\colon \mathbb{R} \to \mathbb{R}$,*

$$\langle L_{N,\beta}, f \rangle = \frac{1}{N} \sum_{i=1}^N f(\lambda_i) \to \int_{-2}^2 f(x) \frac{1}{2\pi} \sqrt{4-x^2} dx, \quad a.s. \tag{8}$$

Here $\langle \mu, f \rangle = \int f(x) d\mu(x)$ for a probability measure μ and a measurable function f.

The convergence of random probability measures $\{L_{N,\beta}\}$ in the above theorem is in fact the almost sure convergence of probability measure-valued random variables. To show that type of convergence, we only need to verify the almost sure convergence of $\langle L_{N,\beta}, f\rangle$ for a suitable class of test functions f. For instance, each condition below is sufficient for the almost sure convergence of random probability measures (on the real line) $\{\xi_n\}$ to a deterministic limit μ:

(i) the probability measure μ is determined by moments, and for any $k = 0, 1, 2, \dots$, the sequence $\{\langle \xi_n, x^k\rangle\}$ converges almost surely to $\langle \mu, x^k\rangle$;

(ii) for any $z \in D$, the sequence $\{\langle \xi_n, (\cdot - z)^{-1}\rangle\}$ converges almost surely to $\langle \mu, (\cdot - z)^{-1}\rangle$, where D is a dense subset in $\{w \in \mathbb{C} : \mathrm{Im}(w) > 0\}$.

2.1 Potential theory

The argument in this subsection is mainly taken from [7]. Let $\beta > 0$ be fixed. We rewrite the joint density of GβE as

$$\begin{aligned} &p_{N,\beta}(\lambda_1, \lambda_2, \dots, \lambda_N) \\ &= \frac{1}{Z_{N,\beta}} \exp\left\{ -\frac{\beta N^2}{2}\Big(-\frac{1}{N^2}\sum_{i\neq j} \log|\lambda_j - \lambda_i| + \frac{1}{N}\sum_{i=1}^N \frac{\lambda_i^2}{2}\Big)\right\} \\ &= \frac{1}{Z_{N,\beta}} \exp\left\{ -\frac{\beta N^2}{2}\Big(-\iint_{x\neq y} \log|x-y| dL_N(x) dL_N(y) + \int V(x) dL_N(x)\Big)\right\}. \end{aligned} \tag{9}$$

Here $L_N = L_{N,\beta} = N^{-1}\sum_{i=1}^N \delta_{\lambda_i}$, and $V(x) = x^2/2$.

Let

$$\begin{aligned} I_V[\mu] &= -\iint \log|x-y| d\mu(x) d\mu(y) + \int V(x) d\mu(x) \\ &= \iint \Big(-\log|x-y| + \frac{1}{2}V(x) + \frac{1}{2}V(y)\Big) d\mu(x) d\mu(y), \end{aligned} \tag{10}$$

be the energy functional defined on the set $\mathcal{P}(\mathbb{R})$ of all probability measures on $\mathbb{R}$. Roughly speaking, the equilibrium of the system should minimize the energy. It follows from a general theory that under some conditions on the potential V, there is a unique probability measure μ_V for which

$$I_V[\mu_V] = \inf_{\mu \in \mathcal{P}(\mathbb{R})} I_V[\mu]. \tag{11}$$

In our considering case, $V(x) = x^2/2$, the minimizer could be shown to be the semicircle distribution which will be denoted by μ_∞,

$$d\mu_\infty(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\mathbf{1}_{\{|x|\le 2\}} dx. \tag{12}$$

Moreover, the following result holds.

Theorem 2.2. *Let*

$$u_{N,\beta}^{(k)}(x_1, \dots, x_k) = \int p_{N,\beta}(x_1, \dots, x_k, \lambda_{k+1}, \dots, \lambda_N) d\lambda_{k+1} \cdots d\lambda_N \tag{13}$$

be the k-point function of GβE. Then for any bounded continuous function ϕ on $\mathbb{R}^k$,

$$\begin{aligned} &\lim_{N\to\infty} \int_{\mathbb{R}^k} \phi(x_1, \dots, x_k) u_{N,\beta}^{(k)}(x_1, \dots, x_k) dx_1 \cdots dx_k \\ &= \int_{\mathbb{R}^k} \phi(x_1, \dots, x_k) d\mu_\infty(x_1) \cdots d\mu_\infty(x_k). \end{aligned} \tag{14}$$

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Then, the expectation and the variance of $\langle L_N, f\rangle$ can be expressed as

$$\begin{aligned}
\mathbb{E}[\langle L_N, f\rangle] &= \int f(x) u^{(1)}_{N,\beta}(x)dx,\\
\mathrm{Var}[\langle L_N, f\rangle] &= \frac{N(N-1)}{N^2}\iint f(x)f(y)u^{(2)}_{N,\beta}(x,y)dxdy + \frac{1}{N}\int f(x)^2 u^{(1)}_{N,\beta}(x)dx\\
&\quad - \Big(\int f(x)u^{(1)}_{N,\beta}(x)dx\Big)^2.
\end{aligned}$$

As a direct consequence of the above theorem, as $N \to \infty$,

$$\mathbb{E}[\langle L_N, f\rangle] \to \langle \mu_\infty, f\rangle, \quad \mathrm{Var}[\langle L_N, f\rangle] \to 0,$$

which implies that $\langle L_N, f\rangle$ converges to $\langle \mu_\infty, f\rangle$ in probability. In fact, for smooth enough function f, we can show that the variance of $\langle L_N, f\rangle$ is of order N^{-2}, from which the almost sure convergence follows.

2.2 Variational formula

This subsection introduces another way to identify the equilibrium of the system based on the so called variational formula. We also fix $\beta > 0$ here. Let $\phi \in C^1(\mathbb{R})$ with ϕ' bounded from below. For $\lambda > 0$ small enough such that $\lambda\phi'(y) > -1$ for all $y \in \mathbb{R}$, we make the change of variables $\lambda_i = y_i + \lambda\phi(y_i)$ in the integral

$$Z_{N,\beta} = \int \exp\Big\{\frac{\beta}{2}\Big(\sum_{i\neq j}\log|\lambda_j - \lambda_i| - N\sum_i V(\lambda_i)\Big)\Big\}d\lambda_1\cdots d\lambda_N.$$

It can be deduced from $\frac{d}{d\lambda}\log Z_{N,\beta}|_{\lambda=0+} = 0$ that (cf. Eq. (2.18) in Johansson (1998) [7])

$$\begin{aligned}
\frac{\beta}{2}N(N-1)\iint \frac{\phi(x)-\phi(y)}{x-y}u^{(2)}_{N,\beta}(x,y)dxdy - \frac{\beta}{2}N^2\int V'(x)\phi(x)u^{(1)}_{N,\beta}(x)dx&\\
+ N\int \phi'(x)u^{(1)}_{N,\beta}(x)dx = 0.& \qquad (15)
\end{aligned}$$

The equation is called a variational formula.

For $z \in \mathbb{C}\setminus\mathbb{R}$, let $\phi(t) = 1/(t-z)$. Since the derivatives of the real part and the image part of ϕ are bounded, the equation (15) also holds for the complex-valued function ϕ. Simplifying it yields

$$\frac{(N-1)}{N}\iint \frac{u^{(2)}_{N,\beta}(x,y)}{(x-z)(y-z)}dxdy + \int \frac{V'(x)u^{(1)}_{N,\beta}(x)}{x-z}dx + \frac{2}{N\beta}\int \frac{u^{(1)}_{N,\beta}(x)}{(x-z)^2}dx = 0. \qquad (16)$$

Let $S(z) = \int (x-z)^{-1}d\mu_\infty(x), z \in \mathbb{C}\setminus\mathbb{R}$, be the Stieltjes transform of μ_∞. Then Theorem 2.2 implies that as $N \to \infty$,

$$\begin{aligned}
\iint \frac{u^{(2)}_{N,\beta}(x,y)}{(x-z)(y-z)}dxdy &\to \iint \frac{d\mu_\infty(x)d\mu_\infty(y)}{(x-z)(y-z)} = S(z)^2,\\
\int \frac{V'(x)u^{(1)}_{N,\beta}(x)}{x-z}dx &= 1 + z\int \frac{u^{(1)}_{N,\beta}(x)}{x-z}dx \to 1 + zS(z),\\
\int \frac{u^{(1)}_{N,\beta}(x)}{(x-z)^2}dx &\to \int \frac{d\mu_\infty(x)}{(x-z)^2} = S'(z).
\end{aligned}$$

Thus, by letting $N \to \infty$ in the variational formula (16), we get an equation

$$S(z)^2 + zS(z) + 1 = 0.$$

Solving it with noting that for Stieltjes transforms, $\mathrm{Im}(S(z)) > 0$ when $\mathrm{Im}(z) > 0$, we get

$$S(z) = -\frac{1}{2}\left(z - \sqrt{z^2 - 4}\right),$$

from which, the density of μ_∞ is derived

$$\mu_\infty(x) = \lim_{\varepsilon \searrow 0} \frac{1}{\pi} \operatorname{Im} S(x + i\varepsilon) = \frac{1}{2\pi}\sqrt{4 - x^2}\mathbf{1}_{|x| \le 2}.$$

It is worth mentioning that a variational formula in a more general form (Eq. (2.18) in Johansson (1998)) is the basis to derive a central limit theorem for $\sum_i f(\lambda_i)$.

2.3 A dynamic version

Consider the following system of stochastic differential equations (SDEs)

$$d\lambda_i(t) = \frac{2}{\sqrt{N\beta}} db_i(t) - \lambda_i(t)dt + \frac{2}{N}\sum_{j \ne i} \frac{dt}{\lambda_i(t) - \lambda_j(t)}, \quad i = 1, 2, \dots, N. \tag{17}$$

Here $\{b_i(t)\}_{i=1}^N$ are independent standard Brownian motions. When $\beta \ge 1$, the equations have a unique strong solution in which the particles are non-collide almost surely (Rogers and Shi (1993) [10]). For $0 < \beta < 1$, some boundary condition need is needed, see Cépa and Lépingle (1997) [4]. It turns out that GβE is the stationary distribution of the above SDEs.

Let

$$X_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i(t)}$$

be the empirical distribution of the particles $\{\lambda_i(t)\}$. For C^2-function f, by Itô's formula

$$\begin{aligned}
\langle X_t^N, f\rangle = \langle X_0^N, f\rangle &+ \int_0^t \left(\iint \frac{f'(x) - f'(y)}{x - y} dX_s^N(x) dX_s^N(y)\right) ds \\
&- \int_0^t \langle X_s^N, xf'(x)\rangle ds + \left(\frac{2}{\beta} - 1\right)\frac{1}{N}\int_0^t \langle X_s^N, f''\rangle ds \\
&+ \frac{2}{\sqrt{\beta}N\sqrt{N}}\int_0^t \left(\sum_{i=1}^N f'(\lambda_i(s)) db_i(s)\right) ds.
\end{aligned} \tag{18}$$

This may be regarded as a dynamic version of the variational formula (15). Based on this formula, a dynamic version of Wigner's semicircle law is derived. Again, $\beta > 0$ is assumed to be fixed.

Theorem 2.3. *Let μ_0 be a probability measure on $\mathbb{R}$. Then we can arrange initial positions for the particles such that the sequence of probability measure-valued processes $\{(X_t^N)_t\}_N$ converges weakly to the limit (μ_t), the unique continuous probability measure-valued function satisfying*

$$\langle \mu_t, f\rangle = \langle \mu_0, f\rangle + \int_0^t \left(\iint \frac{f'(x) - f'(y)}{x - y} d\mu_s(x) d\mu_s(y)\right) ds - \int_0^t \langle \mu_s, xf'(x)\rangle ds, \tag{19}$$

for all $f \in C_b^2(\mathbb{R})$ with $xf'(x)$ bounded. Moreover, as $t \to \infty$, μ_t converges weakly to the semicircle distribution.

Fluctuations around the limit were studied by Israelsson (2001) [6].

2.4 Random tridiagonal matrix model

Once we have a random matrix model $T_{N,\beta}$, it is natural to study the limiting behavior of the empirical distribution via the trace of $(T_{N,\beta})^k, k = 0, 1, 2, \dots$, because

$$\langle L_{N,\beta}, x^k\rangle = \frac{1}{N}\sum_{i=1}^N \lambda_i^k = \frac{1}{N}\operatorname{Tr}((T_{N,\beta})^k) = \frac{1}{N}\sum_{i=1}^N (T_{N,\beta})^k(i, i). \tag{20}$$

A sufficient condition for the almost sure convergence of $\langle L_{N,\beta}, x^k\rangle$ is: the convergence of the expectation $N^{-1}\mathbb{E}[\mathrm{Tr}((T_{N,\beta})^k)]$ and the almost sure convergence of centered random variables

$$\frac{1}{N}\Big(\mathrm{Tr}((T_{N,\beta})^k) - \mathbb{E}[\mathrm{Tr}((T_{N,\beta})^k)]\Big) \to 0, \quad \text{a.s.} \tag{21}$$

Observe that for fixed k, the random variable $(T_{N,\beta})^k(i,i)$ depends locally on entries of $T_{N,\beta}$ near the (i,i)-location. In particular, $(T_{N,\beta})^k(i,i)$ and $(T_{N,\beta})^k(j,j)$ are independent if $|i-j|$ is large enough, which enables us to show the almost sure convergence (21) quite easily, even when β varies.

Let B_u, for $u \ge 0$, be a doubly infinite Jacobi matrix of the form

$$B_u = \begin{pmatrix} \ddots & \ddots & \ddots & & & \\ & \sqrt{u} & 0 & \sqrt{u} & & \\ & & \sqrt{u} & 0 & \sqrt{u} & \\ & & & \ddots & \ddots & \ddots \end{pmatrix} = \sqrt{u} B_1.$$

For $j \in \{1,2,\dots,N\}$, from the local property of $(T_{N,\beta})^k(i,i)$, we see that

$$\mathbb{E}[(T_{N,\beta})^k(i,i)] \approx B_u^k(0,0), \quad u = (N-i)/N,$$

from which, we can deduce that as $N \to \infty$,

$$\begin{aligned} \frac{1}{N}\mathbb{E}[\mathrm{Tr}((T_{N,\beta})^k)] &= \frac{1}{N}\sum_{i=1}^N \mathbb{E}[(T_{N,\beta})^k(i,i)] \\ &\to \int_0^1 B_u^k(0,0)du = B_1^k(0,0)\int_0^1 u^{k/2}du = \frac{B_1^k(0,0)}{k/2+1}. \end{aligned}$$

When k is odd, it is clear that both $\mathbb{E}[\mathrm{Tr}((T_{N,\beta})^k)] = 0$ and $B_1^k(0,0) = 0$. When $k = 2p$ is even, $B_1^{2p}(0,0)$ counts the number of paths from $(0,0)$ to $(2p,2p)$ in which each step is up or down only by 1 unit. The number of such paths is $\binom{2p}{p}$. Consequently,

$$\frac{1}{N}\mathbb{E}[\mathrm{Tr}((T_{N,\beta})^k)] \begin{cases} = 0, & \text{if } k \text{ is odd,} \\ \to \frac{1}{p+1}\binom{2p}{p} =: C_p, & \text{if } k = 2p \text{ is even.} \end{cases}$$

The numbers $\{C_p\}$ are known as Catalan numbers which coincide with the $(2p)$th moments of the semicircle distribution μ_∞. Note that the odd moments of the semicircle distribution are zero. We have just derived the following result which implies Wigner's semicircle law.

Lemma 2.4. *Let $\beta > 0$ be fixed. Then for any $k = 0,1,2,\dots,$ as $N \to \infty$,*

$$\langle L_{N,\beta}, x^k\rangle \to \langle \mu_\infty, x^k\rangle, \quad a.s.$$

Another way to show the convergence of the expectation $\mathbb{E}[\mathrm{Tr}((T_{N,\beta})^k)]$ is to use the following interesting property

$$\frac{1}{N}\mathbb{E}[\mathrm{Tr}((T_{N,\beta})^k)] = \mathbb{E}[(T_{N,\beta})^k(1,1)]. \tag{22}$$

The property is shown as follows. Let $\{v_i\}_{i=1}^N$ be the normalized eigenvectors of $T_{N,\beta}$ corresponding to $\{\lambda_i\}_{i=1}^N$. Let $w_i = |v_i(1)|^2, i = 1,2,\dots,N$. Then as proved by Dumitriu and Edelman (2002) [5], $(w_i)_{i=1}^N$ is independent of the eigenvalues $\{\lambda_i\}_{i=1}^N$ and has the same distribution as that of the vector

$$\left(\frac{\chi_{\beta,1}^2}{\sum_{i=1}^N \chi_{\beta,i}^2}, \dots, \frac{\chi_{\beta,N}^2}{\sum_{i=1}^N \chi_{\beta,i}^2}\right),$$

with $\{\chi_{\beta,i}^2\}_{i=1}^N$ being i.i.d. chi-squared distributed random variables with β degrees of freedom. It follows from the spectral decomposition that

$$(T_{N,\beta})^k(1,1) = \sum_{i=1}^N w_i \lambda_i^k.$$

Then the desired equation (22) is a consequence of a simple calculation

$$\mathbb{E}[(T_{N,\beta})^k(1,1)] = \sum_{i=1}^N \mathbb{E}[w_i\lambda_i^k] = \sum_{i=1}^N \mathbb{E}[w_i]\mathbb{E}[\lambda_i^k] = \frac{1}{N}\sum_{i=1}^N \mathbb{E}[\lambda_i^k] = \frac{1}{N}\mathbb{E}[\mathrm{Tr}((T_{N,\beta})^k)].$$

Since $T_{N,\beta}$ is tridiagonal, $(T_{N,\beta})^k(1,1)$ depends only on some top entries. Looking at top entries, we see that

$$T_{N,\beta} = \frac{1}{\sqrt{N\beta}}\begin{pmatrix} \mathcal{N}(0,2) & \chi_{(N-1)\beta} & & \\ \chi_{(N-1)\beta} & \mathcal{N}(0,2) & \chi_{(N-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \chi_\beta & \mathcal{N}(0,2) \end{pmatrix} \to \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & 1 & 0 & 1 \\ & & \ddots & \ddots & \ddots \end{pmatrix} =: J_{sc}.$$

More precisely, this means that for each $i = 1, 2, \dots,$ as $N \to \infty$,

$$T_{N,\beta}(i,i) \to 0, \quad T_{N,\beta}(i,i+1) = T_{N,\beta}(i+1,i) \to 1,$$

in probability and in L^q for any $q \in [1,\infty)$. Consequently, as $N \to \infty$,

$$\mathbb{E}[(T_{N,\beta})^k(1,1)] \to J_{sc}^k(1,1).$$

Given a Jacobi matrix J,

$$J = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad a_i \in \mathbb{R}, b_j > 0,$$

there is a probability measure μ on $\mathbb{R}$ satisfying

$$\int x^k d\mu(x) = J^k(1,1), \quad k = 0, 1, \dots.$$

The matrix J is regarded as a symmetric operator on ℓ^2 with domain $D_0 = \{x = (x_1, x_2, \dots) : x_k = 0 \text{ for } k \text{ sufficiently large}\}$. It is known that the measure μ is unique, or μ is determined by moments, if and only if the operator J is essentially self-adjoint. In case of uniqueness, the probability measure μ is called the spectral measure of J. A useful sufficient condition for that is $\sum_{n=1}^\infty b_n^{-1} = \infty$.

Note that when J is a finite Jacobi matrix of order N, the spectral measure μ of J which is defined the same as above is unique and is given by

$$\mu = \sum_{i=1}^N |v_i(1)|^2 \delta_{\lambda_i},$$

where $\{\lambda_i\}_{i=1}^N$ are the eigenvalues of J, known to be distinct, and $\{v_i\}_{i=1}^N$ are the corresponding normalized eigenvectors. Spectral measures of GβE, or more precisely, of $T_{N,\beta}$ have been studied by Trinh (2018) [12].

Back to our problem here, for the infinite Jacobi matrix J_{sc}, the corresponding spectral measure is nothing but the semicircle distribution. This correspondence can be shown in several ways: by counting the number of Dyck paths, or by calculating the Stieltjes transform (see Trinh (2018) [12]). Note also that the approach via random matrix model here could provide a rigorous proof of Wigner's semicircle law in case $N\beta \to \infty$.

3 The regime $N\beta \to 2c \in (0,\infty)$

This section introduces several approaches, including heuristic ones, to show the following result.

Theorem 3.1. *As $N \to \infty$ with $N\beta \to 2c \in (0,\infty)$, almost surely, the empirical distribution $L_{N,\beta}$ converges weakly to the probability measure μ_c with density*

$$\mu_c(x) = \sqrt{c}\rho_c(\sqrt{c}x),$$

$$\rho_c(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}\frac{1}{|\hat{f}_c(x)|^2}, \text{ where } \hat{f}_c(x) = \sqrt{\frac{c}{\Gamma(c)}}\int_0^\infty t^{c-1}e^{-\frac{t^2}{2}+ixt}dt.$$

3.1 Potential theory

By some heuristic saddle point argument, the equilibrium of the system in the regime where $N\beta \to 2c$ minimizes the following energy functional

$$\mathcal{E}[\mu] = -\iint \log|x-y|\mu(x)dx\mu(y)dy + \int V(x)\mu(x)dx + \frac{1}{c}\int \log(\mu(x))\mu(x)dx, \tag{23}$$

on the set of probability density functions $\mu(x)$

$$\mu(x) \geq 0, \quad \int \mu(x)dx = 1.$$

Refer to Allez et al. (2012) [1] for more details.

3.2 Variational formula

Since the argument in the previous subsection is heuristic, in order to use the variational formula, we need the following conditions for the 1-point function and the 2-point function: for a suitable class of test functions f, as $N \to \infty$ with $N\beta \to 2c$,

$$\begin{aligned}
&\text{(i)} \quad \int f(x)u^{(1)}_{N,\beta}(x)dx \to \int f(x)d\mu_c(x),\\
&\text{(ii)} \quad \iint f(x)f(y)u^{(2)}_{N,\beta}(x,y)dxdy \to \left(\int f(x)d\mu_c(x)\right)^2.
\end{aligned}$$

Here μ_c is a probability measure. Note that the condition (i) should be equivalent to the convergence of $\{u^{(1)}_{N,\beta}(x)dx\}$ to μ_c. Under (i), the condition (ii) is equivalent to $\mathrm{Var}[\langle L_{N,\beta}, f\rangle] \to 0$. Both conditions can be proved by using the random matrix model and Poincaré's inequality.

Under those conditions, by letting $N \to \infty$ with $N\beta \to 2c$ in the variational formula (16), we obtain an equation for $S_c(z) = \int (x-z)^{-1}d\mu_c(x), z \in \mathbb{C} \setminus \mathbb{R}$,

$$S_c(z)^2 + 1 + zS_c(z) + \frac{1}{c}S'_c(z) = 0. \tag{24}$$

The above equation is solvable, see Allez et al. (2012) [1]. Then an explicit formula for the density of μ_c as in Theorem 3.1 can be derived. The class $\{\mu_c\}_{c>0}$ is an interpolation between the standard Gaussian distribution and the semicircle distribution. To be more precise, as $c \to \infty$, the probability measure μ_c converges weakly to the semicircle distribution, and when $c \to 0+$, ρ_c converges weakly to the standard Gaussian distribution.

3.3 A dynamic version

The SDEs corresponding to this regime is

$$d\lambda_i(t) = \frac{\sqrt{2}}{\sqrt{c}}db_i(t) - \lambda_i(t)dt + \frac{2}{N}\sum_{j\neq i}\frac{dt}{\lambda_i(t)-\lambda_j(t)}, \quad i = 1,2,\dots,N, \tag{25}$$

with initial condition $\lambda_i(t) = x_0, i = 1,2,\dots,N$. Note that the fourth term in the equation (18) now remains when taking the limit. Then Cépa amd Lépingle (1997) [4] proved the following

Theorem 3.2. *The sequence of measure-valued processes $\{(X^N_t)_t\}$ is weakly convergent and the limit (μ_t) is the unique continuous probability measure-valued function satisfying*

$$\begin{aligned}
\langle \mu_t, f\rangle = f(x_0) + \int_0^t \left(\iint \frac{f'(x)-f'(y)}{x-y}d\mu_s(x)d\mu_s(y)\right)ds - \int_0^t \langle \mu_s, xf'(x)\rangle ds\\
+ \frac{1}{c}\int_0^t \langle \mu_s, f''\rangle ds,
\end{aligned} \tag{26}$$

for all $f \in C^2_b(\mathbb{R})$ with $xf'(x)$ bounded.

3.4 Random tridiagonal matrix model

In the regime where $N\beta \to 2c$, by looking at top entries again,

$$\frac{1}{\sqrt{N\beta}}\begin{pmatrix} \mathcal{N}(0,2) & \chi_{(N-1)\beta} & & \\ \chi_{(N-1)\beta} & \mathcal{N}(0,2) & \chi_{(N-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \chi_\beta & \mathcal{N}(0,2) \end{pmatrix} \xrightarrow{d} \frac{1}{\sqrt{2c}}\begin{pmatrix} \mathcal{N}(0,2) & \chi_{2c} & & \\ \chi_{2c} & \mathcal{N}(0,2) & \chi_{2c} & \\ & \ddots & \ddots & \ddots \end{pmatrix} =: \frac{1}{\sqrt{c}} J_c.$$

Here J_c is an infinite Jacobi matrix with independent entries. For convenience in notations, we consider $\hat{T}_{N,\beta} = (\sqrt{N\beta}/\sqrt{2})T_{N,\beta}$ instead of $T_{N,\beta}$.

Let

$$m_k = m_k(N,\beta) = \mathbb{E}[(\hat{T}_{N,\beta})^k(1,1)].$$

Recall that $m_k = 0$, if k is odd. For even k, from formulas for moments of chi distributions, we see that m_k is a polynomial in N and β, and thus, it can be defined for all $N, \beta \in \mathbb{R}$. Some first m_k's are as follows

$$\begin{aligned} m_2 &= 1 - \frac{\beta}{2} + N\frac{\beta}{2}, \\ m_4 &= 3 - 5\frac{\beta}{2} + 3\left(\frac{\beta}{2}\right)^2 + N\left\{5\frac{\beta}{2} - 5\left(\frac{\beta}{2}\right)^2\right\} + N^2 2\left(\frac{\beta}{2}\right)^2, \\ m_6 &= 15 - 32\frac{\beta}{2} + 32\left(\frac{\beta}{2}\right)^2 - 15\left(\frac{\beta}{2}\right)^3 + N\left\{32\frac{\beta}{2} - 54\left(\frac{\beta}{2}\right)^2 + 32\left(\frac{\beta}{2}\right)^3\right\} \\ &\quad + N^2\left\{22\left(\frac{\beta}{2}\right)^2 - 22\left(\frac{\beta}{2}\right)^3\right\} + N^3 5\left(\frac{\beta}{2}\right)^3. \end{aligned}$$

Now, it follows from the convergence in distribution of top entries of $\hat{T}_{N,\beta}$ that as $N \to \infty$ with $N\beta \to 2c$,

$$m_k(N,\beta) = \mathbb{E}[(\hat{T}_{N,\beta})^k(1,1)] \to \mathbb{E}[J_c^k(1,1)].$$

The key point is this approach is the following duality relation

$$m_{2p}(N,\beta) = (-1)^p \left(\frac{\beta}{2}\right)^p m_{2p}\left(-\frac{N\beta}{2}, \frac{4}{\beta}\right). \tag{27}$$

It follows that

$$\mathbb{E}[J_c^{2p}(1,1)] = \lim_{N\beta\to 2c} m_{2p}(N,\beta) = (-1)^p \lim_{\beta\to\infty} (\beta/2)^{-p} m_{2p}(-c,\beta). \tag{28}$$

Identifying the above limit reduces to the problem of considering $\beta \to \infty$ when N is fixed, which can be easily seen from the random matrix model. Indeed, for $N \in \mathbb{N}$, as $\beta \to \infty$,

$$(\beta/2)^{-1/2}\hat{T}_{N,\beta} = \frac{1}{\sqrt{\beta}}\begin{pmatrix} \mathcal{N}(0,2) & \chi_{(N-1)\beta} & & \\ \chi_{(N-1)\beta} & \mathcal{N}(0,2) & \chi_{(N-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \chi_\beta & \mathcal{N}(0,2) \end{pmatrix} \to \begin{pmatrix} 0 & \sqrt{N-1} & & \\ \sqrt{N-1} & 0 & \sqrt{N-2} & \\ & \ddots & \ddots & \ddots \\ & & \sqrt{1} & 0 \end{pmatrix}.$$

Then the following result follows by exchanging N and $-c$, and changing the sign as well,

$$\mathbb{E}[J_c^{2p}(1,1)] = \lim_{N\beta\to 2c} m_{2p}(N,\beta) = A_c^{2p}(1,1),$$

where A_c is an infinite Jacobi matrix

$$A_c = \begin{pmatrix} 0 & \sqrt{c+1} & & \\ \sqrt{c+1} & 0 & \sqrt{c+2} & \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

The spectral measure of A_c is nothing but the probability measure of associated Hermite polynomials, which coincides with $\rho_c(x)dx$, see Askey and Wimp (1984) [2]. For more detailed discussion of this approach, see Shirai and Trinh (2015) [13]. Recall that the almost sure convergence of centered random variables as in (21) is easily shown. Theorem 3.1 has been proved.

4 Concluding remarks

We have introduced several approaches to show the convergence to an equilibrium of GβE in any regime, that is, as $N\beta \to 2c \in (0, \infty]$,

$$\frac{1}{N}\sum_{i=1}^{N} f(\lambda_i) \to \int f(x) d\mu_c(x), \quad \text{almost surely,}$$

for a suitable class of test functions f. The next natural question is about the fluctuation around the limit. Johansson (1998) [7] established a central limit theorem (CLT) for $\sum_{i=1}^{N} f(\lambda_i)$ with explicit formula for the limiting variance by analyzing the joint density of GβE, where the function f is smooth enough. Using the random matrix model, the author (Trinh (2017) [11]) established a CLT for polynomial test functions via a martingale approach in any regime. An extension to C^1 functions whose derivative is of polynomial growth is then done with the help of Poincaré's inequality. An explicit formula for the limiting variance mentioned in the introduction is a consequence of the result for GOE or GUE, because the limiting variance does not depend on β.

References

[1] Allez, R., Bouchaud, J.P., Guionnet, A.: Invariant beta ensembles and the Gauss-Wigner crossover. Physical review letters **109**(9), 094,102 (2012)

[2] Askey, R., Wimp, J.: Associated Laguerre and Hermite polynomials. Proc. Roy. Soc. Edinburgh Sect. A **96**(1-2), 15–37 (1984)

[3] Benaych-Georges, F., Péché, S.: Poisson statistics for matrix ensembles at large temperature. J. Stat. Phys. **161**(3), 633–656 (2015)

[4] Cépa, E., Lépingle, D.: Diffusing particles with electrostatic repulsion. Probab. Theory Related Fields **107**(4), 429–449 (1997)

[5] Dumitriu, I., Edelman, A.: Matrix models for beta ensembles. J. Math. Phys. **43**(11), 5830–5847 (2002)

[6] Israelsson, S.: Asymptotic fluctuations of a particle system with singular interaction. Stochastic Process. Appl. **93**(1), 25–56 (2001)

[7] Johansson, K.: On fluctuations of eigenvalues of random Hermitian matrices. Duke Math. J. **91**(1), 151–204 (1998)

[8] Nakano, F., Trinh, K.D.: Gaussian beta ensembles at high temperature: eigenvalue fluctuations and bulk statistics. J. Stat. Phys. **173**(2), 295–321 (2018)

[9] Ramírez, J.A., Rider, B., Virág, B.: Beta ensembles, stochastic Airy spectrum, and a diffusion. J. Amer. Math. Soc. **24**(4), 919–944 (2011)

[10] Rogers, L.C.G., Shi, Z.: Interacting Brownian particles and the Wigner law. Probab. Theory Related Fields **95**(4), 555–570 (1993)

[11] Trinh, K.D.: Global spectrum fluctuations for gaussian beta ensembles: A martingale approach. Journal of Theoretical Probability (2017)

[12] Trinh, K.D.: On spectral measures of random Jacobi matrices. Osaka J. Math. **55**(4), 595–617 (2018)

[13] Trinh, K.D., Shirai, T.: The mean spectral measures of random Jacobi matrices related to Gaussian beta ensembles. Electron. Commun. Probab. **20**, no. 68, 13 (2015)

[14] Valkó, B., Virág, B.: Continuum limits of random matrices and the Brownian carousel. Invent. Math. **177**(3), 463–508 (2009)