Progress about computer-assisted proof for the stationary solution of Navier–Stokes equation

Xuefeng LIU (Niigata University)*1
Mitsuhiro NAKAO (Waseda University)
Shin'ichi OISHI (Waseda University)

Abstract

As one of the Millennium Prize Problems, the problem of existence and smoothness of the Navier–Stokes equation draws the attention of mathematician from the world. Meanwhile, the verified computing with assistance of computers has proved to be a promising approach to investigate the solution existence to nonlinear equation systems. In this talk, I will report the latest progress about the solution verification for the stationary Navier–Stokes equation over a non-convex 3D domain.

1. Introduction

The verified computing, as a new approach to investigate the solution existence to nonlinear equation systems, is drawing attention of researchers. In the past decades, there have been several fundamental results as the milestones to the objective of the solution verification for non-linear equations; see early work of M. Plum, M. Nakao and S. Oishi [14, 11, 13]. As a success case, the solution verification of Stokes' wave of extreme form is given in [5] by K. Kobayashi.

In this talk, we explain the basic idea of our newly developed method for the purpose of solution verification for the stationary Navier-Stokes equation over a non-convex 3D domain Ω,

$$-\Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f},\ \text{div}\ \mathbf{u} = 0 \text{ in } \Omega, \mathbf{u} = 0 \text{ on } \partial\Omega\,.$$

Here, $\mathbf{f} : \Omega \to \mathbb{R}^3$ is an applied body force, $\mathbf{u} : \Omega \to \mathbb{R}^3$ is the velocity vector and $p : \Omega \to \mathbb{R}$ is the pressure. In addition, symbols Δ, ∇ and $\nabla\cdot$ denote the Laplacian, gradient and divergence operators, respectively.

The solution verification is under the frame of Newton-Kantorovich's theorem along with the quantitative error analysis for the finite element methods. Such technique has been successfully applied to various non-linear equations, for example, the semilinear elliptic equation [16]. For the kernel problems required in applying Newton-Kantorovich's theorem, we take the following schemes.

1) To bound the norm of the inverse of a differential operator, the algorithm based on the fixed-point theorem [17] is utilized; a reformulation of this algorithm can be found in [16].

This work was supported by KAKENHI (18K03411,16H03950).
2000 Mathematics Subject Classification: MSC-35Q30, MSC-65M60.
Keywords: Navier-Stokes equation, verified computing, finite element method, computer-assisted proof.
*1 e-mail: xfliu@math.sc.niigata-u.ac.jp web:http://www.xfliu.org

2) To give the *a priori* error estimation of the projection from solution existing space to finite element spaces, the hypercircle method [9, 3] is generalized to the Stokes problem to deal with the divergence-free condition.

3) The rigorous eigenvalue estimation for differential operators in 3D domain is provided by using the non-conforming finite element method [7].

2. Function spaces and the main theorem

Define function space V by

$$V = \{v \in \left(H_0^1(\Omega)\right)^3 \mid \text{div } v = 0\} , \tag{1}$$

along with inner product and norm

$$(u, v) := \int_\Omega \nabla u \cdot \nabla v \, d\Omega, \quad \|u\|_V := \sqrt{(u, u)} .$$

The dual space of V is denoted by V^*. Define $\mathcal{A} : V \to V^*$ and $\mathcal{N} : V \to V^*$ by

$$\langle \mathcal{A}[u], v\rangle = (\epsilon \nabla u, \nabla v), \quad \langle \mathcal{N}[u], v\rangle = -((u \cdot \nabla)u, v) + (f, v) .$$

Then the Navier-Stokes equation can be formulated as the equation of functional.

$$\mathcal{F} := \mathcal{A} - \mathcal{N}, \quad \mathcal{F}[u] = 0 .$$

2.1. The main tool of computer-assisted solution verification

Below is the main theorem in our algorithm to verify the solution existence.

Theorem 1 (Newton-Kantorovich's theorem)**.** *Given $\hat{u} \in V$, assume $\mathcal{F}'[\hat{u}]$ is regular and the following inequality holds with constant $\alpha > 0$*

$$\|\mathcal{F}'[\hat{u}]^{-1}\mathcal{F}(\hat{u})\|_V \le \alpha .$$

Let $B(\hat{u}, 2\alpha)(\subset V)$ be the closed ball centered at $\hat{u}$ and the radius being 2α. Assume the following inequality holds for an open ball D satisfying $B(\hat{u}, 2\alpha) \subset D$ along with the constant ω,

$$\|\mathcal{F}'[\hat{u}]^{-1}(\mathcal{F}'[v] - \mathcal{F}'[w])\|_{V,V} \le \omega\|v - w\|_V, \quad \forall v, w \in D .$$

If $\alpha\omega \le 1/2$ holds, then $\mathcal{F}[u] = 0$ has a unique solution in $u \in B(\hat{u}, \rho)$, where ρ is given by

$$\rho := \frac{1 - \sqrt{1 - 2\alpha\omega}}{\omega} .$$

To apply the Newton-Kantorovich theorem, the following quantities should be estimated explicitly.

1. Norm estimation for the inverse of $\mathcal{F}'[\hat{u}]$: $\quad \|\mathcal{F}'[\hat{u}]^{-1}\|_{V^*,V} \le K$.

2. Residue error of $\hat{u}$: $\quad \|\mathcal{F}[\hat{u}]\|_{V^*} \le \delta$.

3. Local continuity of $\mathcal{F}'$: $\quad \|\mathcal{F}'[v] - \mathcal{F}'[w]\|_{V,V^*} \le G\|v - w\|_V, \quad \forall v, w \in D$.

Once the quantities K δ G are evaluated, the constant α and ω can be given as

$$\alpha := K\delta, \quad \omega := KG$$

If $K^2G\delta \le 1/2$ holds, then there exists a unique solution of $\mathcal{F}[u] = 0$ in $B(\hat{u}, \rho)$.

Below we show a related theoretical result from Girault-Raviart's book for exploring the solution existence of the Navier-Stokes equation.

Theorem 2. *Theorem 2.2 (Chapter IV) of [1] Let $\mathcal{N}$ and $\|f\|_{V^*}$ be defined by*

$$\mathcal{N} := \sup_{u,v,w\in V} \frac{\int_\Omega (w\cdot\nabla)uv\,d\Omega}{\|u\|_V\cdot\|v\|_V\cdot\|w\|_V}, \quad \|f\|_{V^*} = \sup_{v\in V}\frac{(f,v)}{\|v\|_V}.$$

If $\mathcal{N}\cdot\|f\|_{V^*}/\epsilon^2 < 1$, then the Navier-Stokes equation has unique solution in V.

This theoretical result can be utilized for solution existence in the case that ϵ is not small. In the section of numerical example, we will show an example with small ϵ that this theory fails to draw conclusion while our proposed method works well.

Sub-problems Compared with the non-linear problems already solved by using Newton-Kantorovich's theorem, there are two challenging sub-problems in solving the Navier-Stokes equation.

a) The *a priori* error estimation for the Stokes equation, especially for domain of general shapes in 3D space.

b) The rigorous eigenvalue estimation for differential operator over domain of general shapes.

Preparation: finite element spaces Let us introduce the finite element method spaces to be used, which are defined over a regular subdivision $\mathcal{T}^h$ for Ω.

Discontinuous space X_h of degree d X_h is the set of piecewise polynomial of degree d without the requirement of continuity. Define $\mathbf{X}_h := (X_h)^3$.

Conforming FEM space $U_h(\subset (H^1(\Omega))^3)$ and $V_h(\subset V)$ of degree k.

- Let U_h be the set of piecewise polynomials of degree up to k, which also belongs to $H^1(\Omega)$. Define $\mathbf{U}_h := (U_h)^3$.
- Let $U_{h,0} := \{u_h \in U_h | u_h = 0 \text{ on } \partial\Omega\}$, $\mathbf{U}_{h,0} := (U_{h,0})^3$.
- Let $\mathbf{V}_h$ be the subspace of $\mathbf{U}_{h,0}$ with member function satisfying the divergence-free condition, i.e., $\mathbf{V}_h = \{u_h \in \mathbf{U}_{h,0} \mid \text{div } u_h = 0\} = \mathbf{U}_h \cap V$.

Construction of V_h Generally, it is difficult to construct $\mathbf{V}_h$ directly due to the divergence free condition. We turn to utilize the Scott-Vogelius type FEM space,

$$\mathbf{V}_h = \{\mathbf{v} \in \mathbf{U}_{h,0} \mid (\text{div } \mathbf{v}, \eta_h) = 0 \ \forall \eta_h \in X_h\},$$

where the degree k of V_h and the degree d of X_h satisfy $d = k - 1$.

The Raviart-Thomas FEM space RT_h of degree m The mth order Raviart-Thomas space is defined as follows.

$$RT_h := \{p_h \in H(\text{div};\Omega) \mid p_h|_K = (a+dx, b+dy, c+dz), a,b,c,d \in P^m(K)\} .$$

where $P^m(K)$ denotes the set of polynomials on element K with degree up to m.

In the discussion below, the selection of k, d and m satisfies $d = m = k-1$.

2.2. Sub-problem a): The *a priori* error estimation for the Stokes equation

Let us consider the following Stokes equation over domain Ω.

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f},\ \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \quad \mathbf{u} = 0 \ \text{ on } \partial\Omega .$$

With the function V defined in (1), the weak formulation is: Find $\mathbf{u} \in \mathbf{V}$, s.t.,

$$(\nabla \mathbf{u}, \nabla \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V} . \tag{2}$$

Conforming finite element solutions To solve the problem numerically, let us apply the finite element spaced introduced in §2. Particularly, to have a stable computation, the mesh is generated by using the method proposed by S. Zhang [18].

The weak formulation of Stokes equation in FEM spaces is given by saddle point problem: Find $\mathbf{u}_h \in \mathbf{U}_{h,0}$, $\rho_h \in \mathbf{X}_h$, s.t.,

$$(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) + (\nabla \cdot \mathbf{v}_h, \rho_h) + (\nabla \cdot \mathbf{u}_h, \eta_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{U}_{h,0}, \forall \eta_h \in \mathbf{X}_h .$$

The hypercircle equation (Prage-Synge's Theorem) The hypercircle equation method, also named by Prage-Synge's Theorem [15], has been successfully applied to the Poisson equation for the purpose of the *a priori* error estimation [9, 3]. Here, we introduced an extended version of the hypercircle equation and construct the *a priori* error estimation for the Stokes equation.

Let $\mathbf{u} \in \mathbf{V}$ be the exact solution of the Stokes equation. Take $\mathbf{p} \in H(\text{div};\Omega)^3$, s.t.

$$\nabla \cdot \mathbf{p} + \nabla \phi + \mathbf{f} = 0, \ \text{for certain } \phi \in H^1(\Omega) .$$

Then for any $\mathbf{v} \in \mathbf{V}$, the following hypercircle equation holds,

$$\|\nabla \mathbf{u} - \nabla \mathbf{v}\|^2 + \|\nabla \mathbf{u} - \mathbf{p}\|^2 = \|\mathbf{p} - \nabla \mathbf{v}\|^2 .$$

A priori *error estimation* Define the quantity κ_h by

$$\kappa_h = \max_{\mathbf{f}_h \in \mathbf{X}_h} \min_{\mathbf{p}_h, \mathbf{v}_h} \frac{\|\mathbf{p}_h - \nabla \mathbf{v}_h\|}{\|\mathbf{f}_h\|} .$$

where the minimization of $\mathbf{p}_h, \mathbf{v}_h$ is subject to $\mathbf{v}_h \in V_h$ and

$$\nabla \cdot \mathbf{p}_h + \nabla \phi_h + \mathbf{f}_h = 0 \text{ for certain } \phi_h \in U_h .$$

By utilizing the quantity κ_h, we obtain the *a priori* error estimation for FEM solution to (2): for any $\mathbf{f} \in L_2(\Omega)^3$,

$$\|\nabla \mathbf{u} - \nabla \mathbf{u}_h\| \le C_h \|\mathbf{f}\| \quad (C_h := \sqrt{\kappa_h^2 + C_{0,h}^2}) . \tag{3}$$

Here $C_{0,h}$ is a computable quantity related to the error estimation of the L^2-projection $\pi_h : L^2(\Omega) \to X_h$: for any $u \in H^1(\Omega)$,

$$\|u - \pi_h u\| \le C_{0,h} \|\nabla u\| \qquad (C_{0,h} = O(h)) . \tag{4}$$

Then for $\mathcal{A}^{-1} := \Delta^{-1}/\epsilon$, we have, with $C_h(\epsilon) = C_h/\epsilon$, for any $\mathbf{f} \in L_2(\Omega)^3$,

$$\|\nabla (I - P_h)(\mathcal{A}^{-1}\mathbf{f})\| \le C_h(\epsilon) \|\mathbf{f}\| .$$

2.3. Sub-problem b): Eigenvalue estimation of differential operators

The evaluation of K is to estimate the norm of the inverse of a differential operator, which reduces to solving eigenvalue problems of operators. Since the involved eigenvalue problem is related to non-self-adjoint differential operator $\mathcal{K}$, it is not easy to deal with directly. A choice is to apply the idea of M. Plum to consider $\mathcal{K} \cdot \mathcal{K}^*$ ($\mathcal{K}^*$: the conjugate operator of $\mathcal{K}$). Here we turn to the method proposed by M. Nakao to avoid the eigenvalue estimation [12]. An earlier similar approach can also be found in S. Oishi [13].

Except for the problem of evaluating K, the eigenvalue problem also appears in estimating various constants, for example, the Poincare constant in 3D domain. Generally it is difficult to give lower bound for the eigenvalues. In [9, 7], the finite element methods are adopted to provide eigenvalue bounds in an efficient way. Here, we introduce the case of the Laplacian with homogeneous Dirichlet boundary condition [7].

Eigenvalue problem Find $\lambda \in \mathbb{R}$ and u such that

$$-\Delta u = \lambda u \text{ in } \Omega; \quad u = 0 \text{ on } \partial\Omega. \tag{5}$$

Let V_h^{CR} be the Crouzeix-Raviart FEM space under the discretized homogeneous Dirichlet condition, i.e., every $v_h \in V_h^{\mathrm{CR}}$ has zero integral on each boundary edge of the mesh. Let $\lambda_{h,k}$ be the approximated eigenvalue obtained by solving the eigenvalue problem in V_h^{CR}: Find $\lambda_h \in R$ and $u_h \in V_h^{\mathrm{CR}}$, s.t.,

$$(\nabla u_h, \nabla v_h) = \lambda_h (u_h, v_h), \quad \forall v_h \in V_h^{\mathrm{CR}}.$$

Then, we have lower bounds as follows [7],

$$\lambda_k \geq \frac{\lambda_{h,k}}{1 + 0.3804^2 h^2 \lambda_{h,k}} \quad (k = 1, 2, \cdots, \dim(V_h^{\mathrm{CR}}) \, .$$

For more results about bounding eigenvalues and various error constants, refer to [10, 8, 2, 4, 6].

3. Outline of the computer-assisted solution verification

In this section, with 4 steps, we explain the construction of approximate solution $\hat{u} \in V_h$ and the estimation K, δ and G.

3.1. Step 1: Approximate solution $\hat{u}$

In test problem, the approximation $\hat{u}$ is taken as the approximation of exact solution u. Let $u_h \in U_{0,h}$ be an approximation to u. Since u_h may not satisfy the divergence-free condition, one-step correction is performed to get $\hat{u} \in V_h$.

Find $\hat{u} \in \mathbf{U}_{h,0}$, $\phi_h \in X_h$ such that

$$(\nabla \hat{u}, \nabla v_h) + (\mathrm{div}\hat{u}, \eta_h) + (\mathrm{div} v_h, \phi_h) = (\nabla u_h, \nabla v_h) \quad \forall v_h \in \mathbf{U}_{h,0}, \eta_h \in X_h$$

The solution $\hat{u}$ of the above equation belongs to V_h. For a well approximate solution u_h, it is expected that $\hat{u} \approx u_h$.

3.2. Step 2: Estimation of K

Theorem 3 (Estimation of K [12]; see a compact proof in [16]). *Suppose the following inequalities holds with the constants ν_1, ν_2, ν_3,*

$$\|P_h \mathcal{A}^{-1}\mathcal{N}'[\hat{u}]\mathbf{u}_c\|_V \le \nu_1 \|\mathbf{u}_c\|_V, \ \forall \mathbf{u}_c \in V^\perp, \tag{6}$$
$$\|N'[\hat{u}]\mathbf{u}\|_{V^*} \le \nu_2 \|\mathbf{u}\|_V, \ \forall \mathbf{u} \in V, \tag{7}$$
$$\|N'[\hat{u}]\mathbf{u}_c\|_{V^*} \le \nu_3 \|\mathbf{u}_c\|_{\mathbf{V}}, \ \forall \mathbf{u}_c \in V^\perp. \tag{8}$$

Here, $V^\perp$ is the orthogonal complement space of V_h in $\mathbf{V}$. Assume the operator $P_h(I - \mathcal{A}^{-1}\mathcal{N}'[\hat{u}])|_{V_h} : V_h \to V_h$ is invertible and the following estimation holds along with the constant τ

$$\left\| \left(P_h(I - \mathcal{A}^{-1}\mathcal{N}'[\hat{u}])|_{V_h} \right)^{-1} \right\|_{L(V,V)} \le \tau \, .$$

Define $\kappa := C_h(\epsilon)(\nu_1 \tau \nu_2 + \nu_3)$. If $\kappa < 1$, then we have

$$\|\{\mathcal{F}'[\hat{u}]\}^{-1}\|_{V^*,V} \le K,$$

where

$$K := \left\| \begin{pmatrix} \tau\left(1 + \frac{C_h \nu_1 \tau \nu_2}{1-\kappa}\right) & \frac{\tau \nu_1}{1-\kappa} \\ \frac{C_h \nu_2 \tau}{1-\kappa} & \frac{1}{1-\kappa} \end{pmatrix} \right\|_E ,$$

and $\|\cdot\|_E$ is the Euclidean norm of a matrix.

Below, we show how to estimate the constants ν_1, ν_2, ν_3 and τ.

Estimation of ν_1 Let $w_h := P_h \mathcal{A}^{-1}\mathcal{N}'[\hat{u}]u_c$, Then

$$(\epsilon \nabla w_h, \nabla v_h) = (N'[\hat{u}]u_c, v_h) \quad \forall v_h \in V_h \, .$$

Taking $v_h := w_h$, then we have

$$\epsilon \|\nabla w_h\|^2 \le \nu_3 \|\mathbf{u}_c\|_V \|w_h\| \le \nu_3 \|\mathbf{u}_c\|_V \cdot C_p \|\nabla w_h\| \, .$$

Hence,

$$\nu_1 \le \frac{1}{\epsilon} \nu_3 C_p \, .$$

Here, C_p is the Poincare constant that satisfies

$$\|v\| \le C_p \|\nabla v\| \quad \forall v \in H_0^1(\Omega) \, .$$

Estimation of ν_2, ν_3 By applying the Schwartz inequality to $(\mathcal{N}'[\hat{u}]u, v)$, it is easy to obtain that

$$\nu_2, \nu_3 \le (\sqrt{3}\|\hat{u}\|_\infty + 3\|\nabla \hat{u}\| C_p) \, .$$

Estimation of τ Given $u_h \in V_h$, let us define a mapping $T : V_h \to V_h$, such that $w_h = T u_h$ satisfies,

$$(\nabla w_h, \nabla v_h) = (\nabla u_h, \nabla v_h) - \frac{1}{\epsilon}(N'[\hat{u}]u_h, v_h) \ \ \forall v_h \in V_h \, .$$

Then the constant τ can be characterized by

$$\tau = \max_{w_h \in V_h} \frac{\|T^{-1} w_h\|_V}{\|w_h\|_V} \, .$$

The mapping T can be constructed by considering the following variation problem: Find $w_h \in \mathbf{U}_{h,0}$, $\phi_h \in X_h$, $c \in \mathbb{R}$,

$$(\nabla w_h, \nabla v_h)+(\text{div } w_h, \eta_h)+(\text{div } v_h, \phi_h)+(d, \phi_h)+(c, \eta_h) = (\nabla u_h, \nabla v_h)-\frac{1}{\epsilon}\,(\mathcal{N}'[\hat{u}]u_h, v_h).$$

for all $v_h \in \mathbf{U}_{h,0}$, $\eta_h \in X_h$, $d \in \mathbb{R}$.

3.3. Step 3: Estimation of δ

First, let us seek $\mathbf{p}_h \in \mathbf{RT}_h$ that gives an approximation to $\nabla\hat{u}$. We select $\mathbf{p}_h$ as the minimizer of

$$\min_{\mathbf{p_h}\in\mathbf{RT_h}} \|\mathbf{p}_h - \nabla\hat{u}\| \,.$$

subject to the constraint condition as follows.

$$(\epsilon \text{ div } \mathbf{p}_h - (\hat{u}\cdot\nabla)\hat{u} + \mathbf{f}, \eta_\mathbf{h}) = 0, \quad \forall \eta_h \in X_h \,.$$

With the selection of $\mathbf{p}_h$, we have

$$\langle \mathcal{F}[\hat{u}], v\rangle = \epsilon\,(\nabla\hat{u} - \mathbf{p}_h, \nabla v) + (-\epsilon \;\text{ div } \mathbf{p}_h + (\hat{u}\cdot\nabla)\hat{u} - \mathbf{f}, v)\;.$$

With the error estimation for projector $\pi_h : L^2(\Omega) \to X_h$, it is easy to have

$$\delta = \|\mathcal{F}[\hat{u}]\|_{V^*} \le \epsilon\|\nabla\hat{u} - \mathbf{p}_h\| + C_{0,h}\|(I-\pi_h)\left((\hat{u}\cdot\nabla)\hat{u} - \mathbf{f}\right)\| \,.$$

3.4. Step 4: Estimation of G

Notice that

$$\langle(\mathcal{F}'[v] - \mathcal{F}'[w])u, \tilde{u}\rangle = ((\mathcal{N}'[v-w])u, \tilde{u})$$

and

$$((v-w)\cdot\nabla)u, \tilde{u}) \le \sqrt{3}\|v-w\|_{L^4}\|\nabla u\|\|\tilde{u}\|_{0,4} \le \sqrt{3}C_{4,p}^2\|\nabla(v-w)\|\cdot\|\nabla u\|\cdot\|\nabla\tilde{u}\| \,,$$

where $C_{4,p}$ is defined by

$$C_{4,p} := \max_{v\in H^1(\Omega)} \frac{\|v\|_{L^4}}{\|\nabla v\|} \,.$$

From Plum's result, we have an upper bound of $C_{4,p}$ as $C_{4,p} \le (2\lambda_1)^{-1/4}$. where λ_1 is the first eigenvalue of Laplacian defined in (5). Thus,

$$G \le 2\sqrt{3}(2\lambda_1)^{-1/2} = \frac{\sqrt{6}}{\sqrt{\lambda_1}} \,.$$

4. Numerical examples

4.1. Problem setting

We perform numericial computation for the problem with the following setting.

$$\Omega = ((-1,1)^2 \setminus [-1,0]^2) \times (0,1), \quad \mathbf{f} = (0, (e^{y-0.7}-1)e^{-5(1-y)^2}\sqrt{1-y}, 0), \quad \epsilon = 0.03 \,.$$

The selectoion of $\mathbf{f}$ helps to create the a vortex in the fluid. See graph of $\mathbf{f}$ and the approximate solution in Fig. 1.

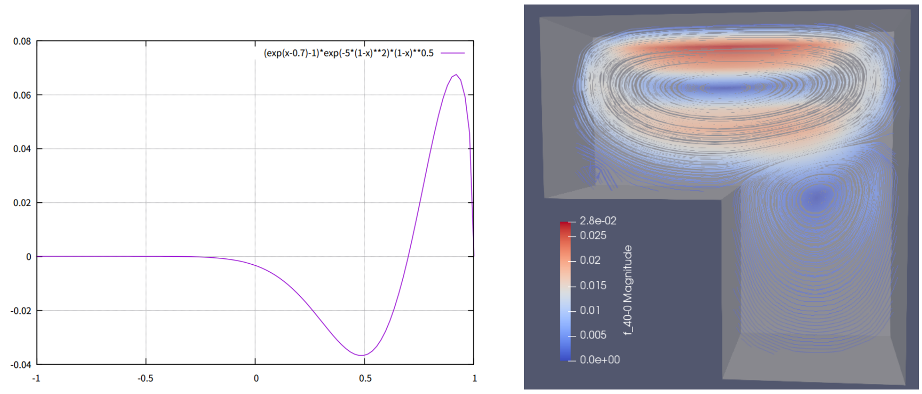

Figure 1: The graph of 2nd part of **f** and the approximate solution

Mesh and FEM spaces

- Mesh $\mathcal{T}^h$: Along x-,y-direction, the longest edge of the domain boundary is divided to $N = 16$ parts, and along the z-direction the edge is divided into $N/2 = 8$ parts. Thus total 512×3 blocks. Then each block is divided into 5 tetrahedra. Finally, each tetrahedron is further divided into 4 sub-tetrahedra by following Zhang's method [18]. Thus the number of elements is $30720 (= 512 \times 3 \times 5 \times 4)$.

- The degree of FEM function spaces is selected as: $d = 1, m = 1, k = 2$.

Values of various quantities used in the solution verification

- The minimal eigenvalue λ_1 of Laplacian with homogeneous boundary condition and the Poincare constant C_p:

$$\lambda_1 > 19.1, \quad C_p = 1/\sqrt{\lambda_1} = 0.2288 \,.$$

- Error constants in *a priori* error estimation: $C_{0,h}$ in (4) and C_h in (3)

$$C_{0,h} = 0.0343, \quad C_h = \frac{1}{\epsilon}\sqrt{\kappa_h^2 + C_{0,h}^2} = \frac{1}{\epsilon}\sqrt{0.0482^2 + 0.0338^2} \leq 1.976 \,.$$

- Estimate of K (norm of inverse operator $\mathcal{F}'[\hat{u}]^{-1}$):

$$\nu_1 = 0.5073, \; \nu_2 = \nu_3 = 0.06651, \; \tau = 2.7082, \; K = 4.1411 \,.$$

- Estimate of residue error of $\mathcal{F}[\hat{u}]$ and local continuity.

$$\delta = 0.000393, \quad G \leq 0.5605 \,.$$

- The condition for Newton-Kantorovich's theorem.

$$\alpha\omega = K^2 \delta G = 0.0038386 < 1/2 \,.$$

Conclusion From the Newton-Kantorovich theorem, without taking the rounding error into account, we can declare the stationary solution existence and uniqueness of the Navier-Stokes equation inside the ball $B(\hat{u}, \rho)$, where

$$\rho = \frac{1 - \sqrt{1 - 2\alpha\omega}}{\omega} = 0.001658\,.$$

Remark 4. *Let us apply Girault-Raviart's theorem to the problem considered here. Since $\mathcal{N}$ is difficult to evaluate, we apply the theoretical upper bound using quantity G.*

$$\mathcal{N} \le G \approx 0.5605, \quad \|f\|_{V^*} \approx 0.0049\,.$$

With $\epsilon = 0.03$, we have

$$\mathcal{N} \cdot \|f\|_{V^*}/\epsilon^2 \le 0.5605 * 0.0049/0.03^2 \approx 3.052 \quad (> 1)\,.$$

Therefore, the solution existence cannot be shown easily by only using Girault-Raviart's theoretical result.

5. TODO problems

In this talk, we propose a realizable method to provide solution verification for the stationary solution of the Navier-Stokes equation. To have a complete proof, the rounding error should be estimated rigorously. However, this is not trivial work, because the matrix involved in the computing is has the dimension about 1 million. Such problem will be attacked in near future with the help of researchers on high-performance computing. Also, larger Reynolds number will bring essential difficult in the solution verification, which is also challenging working in the future research.

Acknowledgement The authors show thanks to Dr. Shinya UCHIUMI for sharing the literature of Scott-Vogelius FEM spaces and mesh generation used in solving the Stokes equation.

References

[1] V Girault and P-A Raviart. *Finite element methods for Navier-Stokes equations: theory and algorithms*, volume 5. Springer Science & Business Media, 2012.

[2] F Kikuchi and X Liu. Estimation of Error Constants Appearing in Non-conforming Linear Triangular Finite Element. *Proc. Comput. Enginnering Conf.*, 12, 2007.

[3] F Kikuchi and H Saito. Remarks on a posteriori error estimation for finite element solutions. *J. Comput. Appl. Math.*, 199:329–336, 2007.

[4] K Kobayashi. On the interpolation constants over triangular elements (in Japanese). *Kyoto Univ. Res. Inf. Repos.*, 1733:58–77, 2011.

[5] K Kobayashi. Computer-assisted Uniqueness Proof for Stokes' Wave of Extreme Form. In *Emerg. Top. Differ. Equations Their Appl.*, pages 54–67. World Scientific, 2013.

[6] K Kobayashi. *On the interpolation constants over triangular elements.* Institue of Mathematics,, Czech Academy of Sciences, 2015.

[7] X Liu. A framework of verified eigenvalue bounds for self-adjoint differential operators. *Appl. Math. Comput.*, 267, 2015.

[8] X Liu and S Oishi. Guaranteed high-precision estimation for P_0 interpolation constants on triangular finite elements. *Jpn. J. Ind. Appl. Math.*, 30:635–652, 2013.

[9] X Liu and S Oishi. Verified eigenvalue evaluation for the Laplacian over polygonal domains of arbitrary shape. *SIAM J. Numer. Anal.*, 51(3):1634–1654, 2013.

[10] X Liu and C You. Explicit bound for quadratic Lagrange interpolation constant on triangular finite elements. *Appl. Math. Comput.*, 2017.

[11] M Nakao. A numerical verification method for the existence of weak solutions for nonlinear boundary value problems. *J. Math. Anal. Appl.*, 164(2):489–507, 1992.

[12] M T Nakao, K Hashimoto, and Y Watanabe. A numerical method to verify the invertibility of linear elliptic operators with applications to nonlinear problems. *Computing*, 75(1):1–14, 2005.

[13] S Oishi. Numerical verification of existence and inclusion of solutions for nonlinear operator equations. *J. Comput. Appl. Math.*, 60(1-2):171–185, 1995.

[14] M Plum. Computer-assisted existence proofs for two-point boundary value problems. *Computing*, 46(1):19–34, 1991.

[15] W Prager and J L Synge. Approximations in elasticity based on the concept of function space. *Quart. Appl. Math*, 5(3):1–21, 1947.

[16] A Takayasu, X Liu, and S Oishi. Verified computations to semilinear elliptic boundary value problems on arbitrary polygonal domains. *Nonlinear Theory Its Appl. IEICE*, 4(1):34–61, 2013.

[17] Y Watanabe, N Yamamoto, and M T Nakao. A numerical verification method of solutions for the Navier-Stokes equations. *Reliab. Comput.*, 5(3):347–357, 1999.

[18] S Zhang. On the divergence-free finite element method for the Stokes equations and the P1 Powell-Sabin divergence-free element. *SIAM J. Numer. Anal.*, pages 1–18.