The Lefschetz properties of graded Artinian Gorenstein algebras

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1. Introduction

The cohomology ring of a topological space is a skew commutative graded ring. There is no distinction between right-ideal and left-ideal, so it is very close to a commutative ring. The subring of a cohomology ring consisting of the homogeneous parts of even degrees is a commutative graded ring. We consider only commutative graded rings.

We are interested in the properties of graded Artinian Gorenstein rings $A = \bigoplus_{i=0}^{d} A_i$. It is well known that graded Artinian Gorenstein rings satisfy an analogue of Poincaré duality and in this sense one can regard them as algebraic analogues of cohomology rings of smooth complex manifolds. It is then an interesting problem to understand which properties of complex manifolds carry over to this purely algebraic setting. For example, cohomology rings of complex manifolds with a Kähler structure satisfy the hard Lefschetz theorem. On the algebraic side, one can ask which graded Artinian Gorenstein rings satisfy something analogous called the strong Lefschetz property.

In 1983 David Rees posed the following problem: For which ideals I is it true that $\mu(J) \leq \mu(I)$ for all ideals $J \supset I$? ($\mu(I)$ denotes the number of generators of the ideal I, see [16].) In an effort to answer this problem, the definition of the Lefschetz property for Gorenstein algebras naturally arose and I spoke about the definition and some properties concerning it at the Japan–US conference for commutative algebra and combinatorics in summer 1985. In my paper [17] I called it the Stanley property rather than the Lefschetz property, since R. Stanley had done a pioneering work in [15] where he used the hard Lefschetz theorem to prove that certain posets have the Sperner property.

In this conference, I spoke about the conjecture which says we could expect that every complete intersection has the strong Lefschetz property over a field of characteristic zero. I had hoped that this drew the attention of many researchers to this problem, but there were no clues to go any further except a few consequences that are derived directly from the definition.

By 1995, except a very few, nobody seemed to be interested in this problem (to prove or disprove the strong Lefschetz property for complete intersections). However, around the turn of the century Migliore and Nagel succeeded to prove that complete intersections with embedding dimension three have the weak Lefschetz property. Their method was to apply the Grauert-Mülich theorem of vector bundles to the syzygy bundle of height three complete intersection ideals. (See [6].)

After that gradually the number of researchers began to increase. In the last 15 years numerous applications have been found, as a result of which the theory of the Lefschetz property is now of interest in its own right. It also has ties to other areas, including combinatorics, algebraic geometry, algebraic topology, commutative algebra and representation theory. The connection between the Lefschetz property and other areas of mathematics are not only diverse, but sometimes surprising, e.g., its ties to the Schur-Weyl duality.

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Migliore-Nagel [11] is a very good introduction to the theory of the strong and weak Lefschetz properties.

2. Definition of the strong Lefschetz property

"The strong Lefschetz property" can be defined for a graded vector space.

Definition 1. Suppose that $V = \bigoplus_{i=0}^{d} V_i$ is a graded vector space over a field K and $f \in \mathrm{End}_K(V)$ is a graded endomorphism of degree one (i.e., $f(V_i) \subset V_{i+1}$). We say that f is a **Lefschetz endomorphism** if the map

$$f^{d-2i} : V_i \to V_{d-i}$$

is bijective for all $i = 0, 1, 2, \dots, [d/2]$. We say that a pair (V, f) of a graded vector space and an endomorphism has the **strong Lefschetz property** if f is a Lefschetz endomorphism. The sequence of integers

$$\dim_K V_0, \dim_K V_1, \dots, \dim_K V_d$$

is called the **Hilbert series** of the vector space V. Sometimes it is written as a polynomial in q

$$a_0 + a_1 q + a_2 q^2 + \cdots + a_d q^d,$$

where $a_i = \dim_K V_i$. d is called the top degree. We apply a similar definition to a degree -1 map $f \in \mathrm{End}_K(V)$ (i.e., $f(V_i) \subset V_{i-1}$) as well as a degree $+1$ map. Unless otherwise specified, we take f to be a degree $+1$ endomorphism.

The next proposition is obvious but important.

Proposition 2. Suppose that $V = \bigoplus_{i=0}^{d} V_i$ is a graded vector space. For the vector space V to afford a strong Lefschetz endomorphism, it is necessary that the Hilbert series be unimodal and symmetric.

Remark 3. When we say that $\left(V = \bigoplus_{i=0}^{d} V_i, f\right)$ has the strong Lefschetz property, we do not exclude the case $V_0 = V_d = 0$. If $d = 2d'$ and if $V_k = 0$ for $\forall k \neq d'$, then $(V, 0)$ has trivially the strong Lefschetz property. Even if $V_d = 0$, we call d the top degree. The half integer (or integer) $d/2$ is called the reflecting degree.

Example 4. Let K be any field, $K[x]$ the polynomial ring in one variable, and put $V = K[x]/(x^d)$. Then we may consider the multiplication map $\times x : V \to V$ defined by $\overline{v} \mapsto \overline{vx}$. The pair $(V, \times x)$ has the strong Lefschetz property. The top degree is $d - 1$.

Example 5. Let K be a field, $\mathrm{char} K = p$, $p = 0$ or $p > c + d - 2$, and $K[x, y]$ the polynomial ring in two variables. Put $V = K[x, y]/(x^c, y^d)$. Let $\times(x + y) : V \to V$ be the multiplication map by $(x + y)$, i.e, $\times(x + y)$ is defined by $\overline{v} \mapsto \overline{v(x + y)}$. Then $(V, \times(x + y))$ has the strong Lefschetz property. (Proof is not so easy as it seems.) The top degree of V is $c + d - 2$.

Example 6. Let P be the set of square-free monomials in the variables $\{x_1, x_2, \dots, x_n\}$, and $P_k = \{S \in P \mid \text{degree } S = k\}$. We have the decomposition $P = \bigsqcup_{k=0}^{n} P_k$. Let f_k be the 01-matrix representing the divisibility relation between P_k and P_{k+1}. Namely the matrix f_k is defined by $f_k = (f_{kSS'})$ with

$$f_{kSS'} = \begin{cases} 1, & \text{if } S \text{ divides } S',\ (S, S') \in P_k \times P_{k+1}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that S, S' are monomials and the rows and columns of f_k are indexed by the sets P_k and P_{k+1} respectively. Let V be the vector space spanned by P over $K = \mathbb{Q}$ and V_k by P_k. Then $V = \bigoplus_{i=0}^{n} V_i$ is a graded vector space and the Hilbert series is $(1+q)^n$, i.e.,

$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}.$$

We may regard f_k as a linear map $f_k : V_k \to V_{k+1}$, so we have a sequence of homomorphisms

$$V_0 \xrightarrow{f_0} V_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} V_n.$$

Let $f = (f_k)_{k=0}^{n-1}$. Then f is a degree one endomorphism $f : V \to V$. (V, f) has the strong Lefschetz property. (Various proofs are known.) The top degree is n.

Proposition 7. Let $V = \bigoplus_{i=0}^{d} V_i$ be a graded vector space. If (V, f) has the strong Lefschetz property, then so does $(V/\ker f, \overline{f})$. The top degree is $d-1$. It follows that, for any integer $k > 0$, $(V/\ker f^k, \overline{f})$ has the strong Lefschetz property. (The top degree $d-k$.)

Proposition 8. Let $V = \bigoplus_{i=0}^{d} V_i$ be a graded vector space and suppose that (V, f) has the strong Lefschetz property. Suppose that $g \in \mathrm{End}(V)$ is a degree k map and $fg = gf$ and $\mathrm{rank}(f^{i+k} : V \to V) = \mathrm{rank}(f^i g : V \to V)$ for all $i = 0, 1, \ldots$. Then $(V/(\ker g), \overline{f})$ has the strong Lefschetz property.

It is possible to characterize the strong Lefschetz property in terms of the Lie algebra $\mathfrak{sl}(2)$ as follows:

Theorem 9. We assume that the ground field is of characteristic zero. TFAE

1. A graded vector space (V, f) has the strong Lefschetz property.

2. There exists a degree -1 map $g \in \mathrm{End}(V)$ such that, if $h = [f, g]$, then $\{f, g, h\}$ is an $\mathfrak{sl}(2)$-triple, and V_k is the eigenspace of h with eigenvalue $2k - d$.

(Proctor [14] introduced the concept of $\mathfrak{sl}(2)$-poset to prove the Sperner property for certain posets.) This characterization enables us to prove that the strong Lefschetz property is preserved by taking a tensor product, which can be stated as follows:

Theorem 10. Suppose that $\left(U = \bigoplus_{i=0}^{c} U_i, f\right)$ and $\left(V = \bigoplus_{j=0}^{d} V_j, g\right)$ have the strong Lefschetz property. Then the tensor product

$$(W = U \otimes_K V, f \otimes 1 + 1 \otimes g)$$

has the strong Lefschetz property. The grading of W is given by

$$W = \bigoplus_{k=0}^{c+d} \left(\bigoplus_{i+j=k} U_i \otimes_K V_j \right).$$

3. Artinian Gorenstein algebras

We assume that K is a field of characteristic zero. Let $R = K[x_1, x_2, \dots, x_n] = \bigoplus_{i \geq 0} R_i$ be the polynomial ring. (We assume that the degrees of the variables are all 1.) For $g = g(x), f = f(x) \in R$, we define "$g \circ f \in R$", an operation of g on R, as follows:

$$g \circ f = g\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n}\right) f(x)$$

The following is known as the Double Annihilator Theorem of F. S. Macaulay.

Theorem 11. Let $F \in R_d$. We denote by $\mathrm{Ann}(F)$ the annihilator of F:

$$\mathrm{Ann}(F) = \{f \in R | f \circ F = 0\}.$$

Then $\mathrm{Ann}(F)$ is an ideal in R and $R/\mathrm{Ann}(F)$ is an Artinian Gorenstein algebra. Conversely if an Artinian algebra $A = R/I$ is Gorenstein, then there exists $F \in R$ such that $I = \mathrm{Ann}(F)$. For the time being, this may be thought of as the definition of a **graded Artinian Gorenstein algebra**. F is called a **Macaulay dual generator** of A. F is unique up to a scalar multiple. (See [13].)

By the definition of "$\circ$", $R \circ F$ is the graded vector space spanned by the partial derivatives of F. (The top degree is the degree of F.)

Remark 12. Let $F \in R_d$. It is easy to prove that the graded vector spaces $R_j \circ F$ and $R_{d-j} \circ F$ have the same dimension. Thus $R \circ F$ has the symmetric Hilbert series. (This can be proved without knowledge of Gorenstein algebras.)

For $F \in R_d$ let $V = R \circ F = \bigoplus_{i=0}^{d} (R_{d-i} \circ F)$, $D = a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n}$, where $a_i \in K$. Then we have a pair (V, D) of a graded vector space V and a degree -1 endomorphism D. We regard V as a graded subspace of R. Similarly if l is a linear form in R, then we have a pair $(A, \times l)$ of a graded vector space $A = R/\mathrm{Ann}(F)$ and a map $\times l : A \to A$ of degree one.

Definition 13. We say that a Gorenstein algebra $A = \bigoplus_{i=0}^{d} A_i$ has the **strong Lefschetz property** if there exists a linear form $l \in R_1$ such that $(A, \times l)$ has the strong Lefschetz property. I.e., $\times l^{d-2i} : A_i \to A_{d-i}$ is bijective for $i = 0, 1, \dots, [d/2]$. Let F be the Macaulay dual generator for A. Then this is equivalent to the strong Lefschetz property of the pair $(R \circ F, D)$. Note D is a degree -1 map. We say that A has the **weak Lefschetz property** if there exists a linear element $l \in A_1$ such that the multiplication $\times l : A_i \to A_{i+1}$ is either injective or surjective for $i = 0, 1, \dots, d-1$.

Suppose that we have an Artinian Gorenstein algebra R/I. Then I contains high powers of all variables. Since $\mathrm{Ext}^n_R(R/I, R) \cong R/I$, it is possible to express I as

$$I = (x_1^{d_1}, x_2^{d_1}, \dots, x_n^{d_n}) : g,$$

for some homogeneous form $g \in R$. Thus we can use Proposition 8 and Theorem 10 to obtain the following result.

Theorem 14. Suppose that $I = (x_1^{d_1}, x_2^{d_1}, \dots, x_n^{d_n}) : g$. If g is general enough, then the algebra $A = R/I$ has the strong Lefschetz property. Thus "almost all" Artinian Gorenstein algebras have the strong Lefschetz property.

In the next section we exhibit Gorenstein algebras which fail to have the strong Lefschetz property.

4. Higher Hessians, a criterion for the strong Lefschetz property

Let $R = K[x_1, \ldots, x_n]$. Suppose that $A := R/I$ a Gorenstein algebra and let

$$h_0, h_1, \ldots, h_d$$

be the Hilbert series of A. Let $F \in R_d$ be the Macaulay dual generator for A. Note that $h_1 = h_{n-1} = n$ if and only if the partial derivatives $F_{x_1}, \ldots, F_{x_n}$ are linearly independent. We want to define the kth Hessian matrix. Assume that $k \leq [n/2]$. Let $\{\alpha_1, \ldots, \alpha_N\} \subset R_k$ be a set of monomials such that they are a basis for A_k ($N = \dim_K A_k = h_k$). We use the notation $F_\alpha := \alpha \circ F$. (For example, $F_{x_1 x_2} = \frac{\partial^2}{\partial x_1 \partial x_2} F$.) We call

$$\mathrm{Hess}_F^k = (F_{\alpha_i \alpha_j})_{1 \leq i,j \leq N}$$

the kth Hessian matrix. As a matrix it depends on the set of monomials $\{\alpha_j\}$ but the vanishing of the determinant is independent of the choice of monomials. The determinant of Hess_F^k is denoted by hess_F^k. The first Hessian determinant is the same as the Hessian in the normal sense if F involves properly n variables. Since hess_F^k is a polynomial in $x_1, x_2, \ldots, x_n$, we sometimes write $\mathrm{hess}_F^k = \mathrm{hess}_F^k(x_1, x_2, \ldots, x_n)$ as necessary.

Theorem 15. . For a linear form $l = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \in R, a_i \in K$ to be a Lefschetz element of A, it is necessary and sufficient that

$$\mathrm{hess}^k(a_1, a_2, \ldots, a_n) \neq 0, \text{ for } k = 1, 2, \ldots, [n/2].$$

Consequently the following conditions are equivalent. (See [10].)

1. $R/\mathrm{Ann}(F)$ fails to have the strong Lefschetz property.
2. hess_F^k vanishes identically for some k.

To find a Gorenstein algebra which fails the SLP is the same as finding a homogeneous form such that one of higher Hessian determinants (including Hessian in the normal sense) vanish identically. This problem goes back to Hesse's claim made in 1850s. Otto Hesse claimed twice in 1851 and 1855, if the Hessian (in the normal sense) vanishes identically, a variable can be eliminated by means of a linear transformation of the variables. However this is not true. There is a counter example, a quinary cubic due to Gordan and Noether. Nonetheless Hesse's claim is true in 4 variables.

The following was proved by Gordan-Noether in [5]. See also [2], [18], [19].

Theorem 16. 1. If a homogeneous polynomial $F \in \mathbb{C}[w, x, y, z]$ has vanishing Hessian (in the normal sense) then by a linear change of variables, F can be transformed so that $F \in \mathbb{C}[x, y, z]$.

2. Suppose that a form $F \in \mathbb{C}[v, w, x, y, z]$ properly involves 5 variables. If the Hessian (in the normal sense) vanishes identically, then by means of a linear transformation of the variables F can be transformed so that $F \in \mathbb{C}[v, w][G]$, where $G = v^2 x + vwy + w^2 z$. (F is determined as a solution of a certain system of linear partial differential equations.)

Recall that the unimodality of the Hilbert series is necessary for A to have the strong Lefschetz property. R. Stanley constructed a Gorenstein algebra in 13 variables which has Hilbert series (1 13 12 13 1). It was obtained as $R/\mathrm{Ann}(F)$, where $F = M_1x_1 + M_2x_2 + \cdots + M_{10}x_{10}$ and where $\{M_1, \ldots, M_{10}\} \subset \mathbb{C}[u_1, u_2, u_3]_3$ are the set of all monomials of degree three.

Remark 17. If $A = \bigoplus_{i=0}^{d} A_i$ has a non-unimodal Hilbert series, then A fails the SLP. Examples of Gorenstein algebras with unimodal Hilbert series and failing the strong Lefschetz property can be found in Gondim [4].

5. Some classes of Gorenstein algebras which can be proved to have the strong Lefschetz property

As before we work on the polynomial ring $R = K[x_1, \ldots, x_n]$ and a graded Artinian Gorenstein algebra $A = R/I$.

Definition 18. The dimension $\dim A_1$ is called the **embedding dimension** of a Gorenstein algebra A.

Theorem 19.

1. Assume that char $K = 0$ or greater than the top degree. Gorenstein algebras with embedding dimension 2 have the strong Lefschetz property. (Iarrobino)
2. Assume that char $K = 0$. Complete intersections with embedding dimension 3 have the weak Lefschetz property. (Migliore-Nagel)
3. Assume that char $K = 0$. If A has embedding dimension 4, and top degree $d \leq 4$, then A has the strong Lefschetz property. This follows from the result of Gordan-Noether. H. Ikeda [9] constructed a Gorenstein algebra in 4 variables with top degree 5 which fails the strong Lefschetz property.

Conjecture 20. $A = K[x_1, x_2, \ldots, x_n]/I$, where $I = (f_1, f_2, \ldots, f_n)$. Assume that characteristic of K is zero or greater than the top degree. Then we conjecture that A has the strong Lefschetz property.

Theorem 21 (Flat Extension Theorem)**.** Suppose that A is a flat extension of B with fiber C. Suppose that both B and C have the SLP. Then A has the SLP. ("SLP" is used for "strong Lefschetz property.")

Corollary 22. The SLP is preserved by a simple extension. I.e., if B has the SLP, then $A = B[z]/(z^d + a_1z^{d-1} + \cdots + a_{d-1}z + a_d)$, with $a_i \in A_i$, has the SLP. If l is a strong Lefschetz element of B, then $l + tz$ is a strong Lefschetz element of A for $\exists t \in A_1$.

Example 23. Let $A = R/(e_1, e_2, \ldots, e_n)$, where e_j is the elementary symmetric polynomial of degree j. Then A has the SLP. (This is the cohomology ring of a flag variety over $\mathbb{C}$, so the hard Lefschetz theorem applies. The point here is that we can prove the SLP of this algebra algebraically using the flat extension theorem.) If $K = \mathbb{R}$, we have

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n \text{ is a Lefschetz element} \Leftrightarrow \prod_{i<j}(a_i - a_j) \neq 0.$$

(Generally speaking if A has the strong Lefschetz property, then a general linear form is a Lefschetz element.)

Example 24. Let $p_k = x_1^k + x_2^k + \cdots + x_n^k$. (So p_k is the power sum of degree k.) Let $A = R/(p_i, p_{i+1}, \ldots, p_{i+n-1})$. Then A has the SLP. (Use the flat extension theorem.)

6. Applications

6.1. The space of square-free monomials

A **partition** of a positive integer λ is a way to express it as a sum of positive integers:

$$\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_d. \tag{1}$$

Two partitions are regarded as the same if the components are the same up to permutation. In this sense it is often denoted by a decreasing sequence of integers as $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d)$, $\lambda = \sum \lambda_j$, but we use the expression (1) as well as the expression in the non-increasing order. In any case when we talk about the dual partition, we have to arrange the terms in the decreasing order. Often Young diagrams of n boxes are used to denote a partition of a positive integer n.

Note that the Hilbert series of an algebra A is a partition of the integer $\dim A$. On the other hand a nilpotent matrix $L \in \mathrm{GL}(A)$ decomposes into Jordan blocks over a suitable vector space basis of A:

$$L = J_1 \dotplus J_2 \dotplus \cdots \dotplus J_r,$$

where J_i is a Jordan cell with diagonal entries zero and superdiagonal entries 1. Thus the Jordan decomposition of a nilpotent matrix is determined by a partition of the size of the matrix.

Remark 25. Suppose that $A = \bigoplus A_j$ is a graded vector space, and L is a degree $+1$ map. Then L is nilpotent. Suppose that $L = L_1 \dotplus L_2 \dotplus \cdots \dotplus L_r$ is the Jordan decomposition. Then L is a strong Lefschetz endomorphism of A if and only if $|L| = |L_1| + |L_2| + \cdots + |L_r|$ is the dual to the partition given by the Hilbert series of A. ($|L| \stackrel{\text{def}}{=} \text{size } L$.)

Example 26. Consider $A = K[x, y, z]/(x^2, y^2, z^2)$. Then A has the strong Lefschetz property with Lefschetz element $l = x + y + z$. The Hilbert series of A is $(1\ 3\ 3\ 1)$. The Jordan decomposition of the endomorphism $\times l : A \to A$ is given by $(4\ 2\ 2)$, the dual to $(1\ 3\ 3\ 1) = (3\ 3\ 1\ 1)$.

Let S_n be the symmetric group of n letters. It is well known that the isomorphism types of irreducible representations of S_n are parametrized by the Young diagrams of n boxes, which is in one-to-one correspondence with the partition of the integer n. Thus it is possible to write V^λ for an S_n-irreducible module corresponding to the partition λ. "Specht polynomials" can be used to construct typical irreducible S_n-modules.

Example 27. Consider S_4. There are 5 Young diagrams of 4 boxes. These are

$$\lambda = (4), (3\ 1), (2\ 2), (2\ 1\ 1), (1\ 1\ 1\ 1).$$

So there are 5 irreducible representations of S_4. Vector spaces V^λ, as S_4-modules can be constructed as follows:

$$\begin{aligned}
V^{(4)} &= \langle 1 \rangle \\
V^{(3\ 1)} &= \langle x_1 - x_2, x_1 - x_3, x_1 - x_4 \rangle \\
V^{(2\ 2)} &= \langle (x_1 - x_2)(x_3 - x_4), (x_1 - x_3)(x_2 - x_4) \rangle \\
V^{(2\ 1\ 1)} &= \langle (x_1 - x_2)(x_1 - x_3)(x_2 - x_3), (x_1 - x_2)(x_1 - x_4)(x_2 - x_4), \\
&\qquad (x_1 - x_3)(x_1 - x_4)(x_3 - x_4) \rangle \\
V^{(1\ 1\ 1\ 1)} &= \langle \prod_{1 \leq i < j \leq 4} (x_i - x_j) \rangle
\end{aligned}$$

(S_n permutes the variables. $\langle * * * \rangle$ denotes the vector space spanned by $* * *$.)

Let $A = K[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$. The set of square-free monomials can be a basis for A. We look at A as an S_n-module and we want to decompose A into irreducible S_n-modules. Since the SLP is preserved by tensor product, A has the SLP with $l = \sum x_j$ as a strong Lefschetz element. Since l is symmetric, A/lA is an S_n-module. Furthermore if $V \subset A$ is any irreducible homogeneous S_n-module, then the image lV is either 0 or isomorphic to V itself as an S_n-module. Hence the decomposition of A as an S_n-module is determined by the irreducible decomposition of A/lA.

Theorem 28. Let $A = K[x_1, \dots, x_n]/(x_1^2, \dots, x_n^2)$. (Assume characteristic of K is 0.) Then A/lA and A decompose as S_n-modules as follows (see [8]):

(1) The case $n = 2m$,

$$\begin{aligned} A/lA &\cong V^{(m,\, m)} \oplus V^{(m+1,\, m-1)} \oplus \cdots \oplus V^{(2m,\, 0)} \\ A &\cong 1 \cdot V^{(m,\, m)} \oplus 3 \cdot V^{(m+1,\, m-1)} \oplus \cdots \oplus (2m+1) \cdot V^{(2m,\, 0)} \end{aligned}$$

(2) The case $n = 2m + 1$,

$$\begin{aligned} A/lA &\cong V^{(m+1,\, m)} \oplus V^{m+2,\, m-1)} \oplus \cdots \oplus V^{(2m+1,\, 0)} \\ A &\cong 2 \cdot V^{(m+1,\, m)} \oplus 4 \cdot V^{(m+2,\, m-1)} \oplus \cdots \oplus 2(m+1) \cdot V^{(2m+1,\, 0)} \end{aligned}$$

6.2. The monomial complete intersections of uniform degree

Let W be an S_n-module. Let $Y^\lambda(-)$ be the Young symmetrizer corresponding to the Young diagram λ of n boxes. We may regard $Y^\lambda(W)$ as the functor to "extract the isotypic component of W," consisting of all irreducible modules each of which is isomorphic to U_λ. (We write U_λ for the isomorphism type of S_n-module corresponding to λ.) Thus it is possible to write

$$W = \bigoplus_{|\lambda|=n} Y^\lambda(W).$$

Let $W = \left(K^d\right)^{\otimes n}$ be the n-fold tensor of K^d. $Y^\lambda(W)$ is a direct sum of copies of U_λ. So $Y^\lambda(W)$ is known if we know the multiplicity of U_λ. The multiplicity is in fact equal to the dimension of the irreducible $\mathrm{GL}(n)$-module V_λ. We have the tensor representation of the general linear group

$$\phi : \mathrm{GL}(d) \to \mathrm{GL}(W).$$

According to the Schur-Weyl duality, ϕ decomposes into the direct product of

$$\phi_\lambda : \mathrm{GL}(d) \to \mathrm{GL}(Y^\lambda(W)),$$

where λ runs over the partitions of n with at most d parts. Furthermore we have

$$Y^\lambda(W) = U_\lambda \otimes V_\lambda,$$

where U_λ is the irreducible S_n-module and V_λ is the irreducible $\mathrm{GL}(d)$-module determined by λ.

Next we want to see what happens if we identify $W = \left(K^d\right)^{\otimes n}$ with the Artinian Gorenstein algebra

$$A(d) := \mathbb{C}[x_1, x_2, \dots, x_n]/(x_1^d, x_2^d, \dots, x_n^d).$$

For a positive integer m, we use the notation $[m]_q$ for the polynomial

$$[m]_q = \frac{1 - q^m}{1 - q} = 1 + q + \cdots + q^{m-1}.$$

Theorem 29. Let $\lambda = (\lambda_1, \ldots, \lambda_d)$, $n = \sum_{j=1}^{d}, \lambda_j$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0$. The Hilbert series of $Y^\lambda(A(d))$ is given by

$$(\dim U_\lambda) q^{\lambda_2 + 2\lambda_3 + \cdots + (d-1)\lambda_d} \prod_{1 \leq i < j \leq d} \frac{[\lambda_i - \lambda_j + j - i]_q}{[j-i]_q}.$$

The graded vector space $Y^\lambda(A(d))$ has the strong Lefschetz property with $x_1 + \cdots + x_n$ as a strong Lefschetz element. (For details see [7] Chapter 9, [12].)

Corollary 30. (We use the same notation as above.) Let $\phi_\lambda : \mathrm{GL}(d) \to \mathrm{GL}(V_\lambda)$ be the irreducible representation of the general linear group. Let $M \in \mathrm{GL}(d)$ be the Jordan matrix with a single eigenvalue a (with superdiagonal entries 1.) Then $\phi_\lambda(M)$ has single eigenvalue a^n and the decomposition into Jordan cells is given as the dual partition to the Hilbert series

$$\prod_{1 \leq i < j \leq d} \frac{[\lambda_i - \lambda_j + j - i]_q}{[j-i]_q}.$$

(Note that this is a polynomial in q. The sum of the coefficients of q is equal to the dimension of V_λ. Thus this is regarded as a partition of the integer $\dim V_\lambda$.)

Remark 31. It is known that the tensor space $(K^d)^{\otimes n}$ decomposes into irreducible $\mathrm{GL}(d)$-modules, and these modules are parametrized by the partitions of n with at most d parts. Let V_λ be the irreducible $\mathrm{GL}(d)$-module corresponding to the partition λ of n with at most d parts. Then the Weyl dimension formula says that

$$\dim V_\lambda = \prod_{1 \leq i < j \leq d} \frac{\lambda_i - \lambda_j + j - i}{j - i},$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0$, $\sum_{j=1}^{d} \lambda_j = n$. The polynomial $h(q) := \prod_{1 \leq i < j \leq d} \frac{[\lambda_i - \lambda_j + j - i]_q}{[j-i]_q}$ is called the q-analog of the Weyl dimension formula. That $Y^\lambda(A(d))$ has the strong Lefschetz property shows that the coefficients of the polynomial $h(q)$ is a symmetric unimodal sequence of integers.

6.3. The ring of invariants of Artinian algebras by a reflection group

With the identification

$$\left(K^d\right)^{\otimes n} \cong A := K[x_1, \ldots, x_n]/(x_1^d, \ldots, x_n^d),$$

the subspace of the LHS spanned by the symmetric tensors corresponds to the ring of invariants of A by the symmetric group S_n. The ring of invariants A^{S_n} can be described as follows:

$$A^{S_n} = K[e_1, e_2, \ldots, e_n]/(p_d, p_{d+1}, \ldots, p_{d+n-1}),$$

where $p_j = \sum_{i=1}^{n} x_i^j$, the power sum symmetric polynomial of degree j. It is implied that the set of n symmetric power sums of consecutive degrees is a complete intersection.

Remark 32. The following conditions are equivalent.

1. $K[x_1, x_2, \ldots, x_n]/(p_d, p_{d+1}, \ldots, p_{d+n-1})$ is Artinian.
2. $K[p_1, p_2, \ldots, p_n]/(K[p_1, p_2 \cdots, p_n] \cap (p_d, p_{d+1}, \ldots, p_{d+n-1}))$ is Artinian.

This raises the following intriguing problem:

Problem 33. For which n-sets of integers $\{i_1, i_2, \cdots, i_n\}$ is the algebra $R/(p_{i_1}, p_{i_2}, \ldots, p_{i_n})$ Artinian? The same question can be asked for complete symmetric polynomials. (The complete symmetric polynomial of degree j is the sum of all monomials of degree j.)

In the special case when $n = 3$ of the above problem, we have the following conjectures.

Conjecture 34. Let $R = K[x, y, z]$ be the polynomial ring in 3 variables over a field K. Let $p_j, h_j \in R_j$ be the power sum and complete symmetric polynomial of degree j respectively. Let $A = \{a, b, c\}$ be a set of integers with $a < b < c$.

1. Assume $\gcd(a, b, c) = 1$. Then $R/(p_a, p_b, p_c)$ is Artinian if and only if $abc \equiv 0 \bmod 6$.

2. $R/(h_a, h_b, h_c)$ is Artinian if and only if (i) $abc \equiv 0 \bmod 6$, (ii) $\gcd(a+1, b+1, c+1) \equiv 1$ and (iii) for all $t \in \mathbb{N}$ with $t > 2$, there exists $d \in \{a, b, c\}$ such that $d + 2 \not\equiv 0, 1 \bmod t$.

Conca, Krattenthaler and Watanabe wrote these conjectures in [1]. This problem has a very broad generalization for sequences of polynomials defined by the same recursive formula as the power sums and complete symmetric polynomials: $p_{d+1} = \sum_{k=1}^{n}(-1)^{k-1}p_{d-k+1}e_k$ and $h_{d+1} = \sum_{k=1}^{n}(-1)^{k-1}h_{d-k+1}e_k$. Moreover it is related to the famous theorem called the Skolem-Mahler-Lech Theorem. See [3].

Problem 35. The above Conjecture 34 is not easy even in the case $a = 1$. Then the problem asks, in the power sum case, for which integers b, c is the ideal $(x + y + z, x^b + y^b + z^b, x^c + y^c + z^c)$ a complete intersection in $K[x, y, z]$. Since there is a linear form in the generators, it decreases the number of variables by 1. Furthermore we can dehomogenize the generators. This also decreases the number of variables by 1. So the problem is actually a problem in the polynomial ring in one variable. The problem reduces as follows: In the polynomial ring $K[t]$ in one variable, the two polynomials

$$t^b + 1 + (-1 - t)^b, t^c + 1 + (-1 - t)^c$$

do not have a common factor if and only if $bc \equiv 0 \bmod 6$. Furthermore one can conjecture that the polynomial $(1 + t)^n + 1 + t^n$ is irreducible over the field of rational numbers if n is a multiple of 6. I asked some number theorists about this problem, but we have not found a proof or a counter example.

Problem 36. In the following problems we are concerned with Artinian Gorenstein algebras and its Macaulay dual generator in the polynomial ring $R = K[x_1, \ldots, x_n]$. Assume that the characteristic of K is zero or greater than the top degree.

1. Suppose that a homogeneous form F is a symmetric polynomial (in the sense it is an invariant of the symmetric group). Prove that $A = R/\mathrm{Ann}(F)$ has the strong Lefschetz property.

2. Suppose that a homogeneous form F is an alternate polynomial (in the sense it is a semi-invariant of the symmetric group S_n.) Prove that $A = R/\mathrm{Ann}(F)$ has the strong Lefschetz property.

3. Suppose that $I = (f_1, \dots, f_n)$ is a homogeneous complete intersection in n variables. Suppose S_n acts on the polynomial ring $K[x_1, \dots, x_n]$ and it permutes the polynomials $f_1, \dots, f_n$ in the same way it permutes the variables $x_1, \dots, x_n$. Prove that $A = R/I$ has the strong Lefschetz property.

4. Suppose that F is a homogeneous form in three variables. Prove that $A = R/\mathrm{Ann}(F)$ has the strong Lefschetz property.

We can look at Conjecture 34 from a different viewpoint as follows:

Problem 37. Let R be the polynomial ring in n variables.

(1) Suppose that an infinite sequence $u_0, u_1, \dots,$ is defined by linear recurrence relation:

$$u_k = \alpha_1 u_{k-1} + \alpha_2 u_{k-2} + \cdots + \alpha_n u_{k-n},$$

where $\alpha_i \in R_i$. (We can choose $u_0, u_1 \dots, u_{n-1}$ and $\alpha_1, \alpha_3, \dots, \alpha_n$ arbitrarily.)

For what n-sets $\{i_1, i_2, \dots, i_n\} \subset \mathbb{N}$, is the algebra $R/(u_{i_1}, u_{i_2}, \dots, , u_{i_n})$ Artinian?

(2) Let $n = 3$, $R = K[x, y, z]$, $\alpha_1 = x+y+z, \alpha_2 = -(xy+xz+yz), \alpha_3 = xyz$. Define u_k by the linear recurrence relation:

$$u_k = \alpha_1 u_{k-1} + \alpha_2 u_{k-2} + \alpha_3 u_{k-3}.$$

For which initial polynomials $(u_0, u_1, u_2) \in R_0 \times R_1 \times R_2$, is it true that

$$R/(u_a, u_b, u_c) \text{ is Artinian} \Leftrightarrow \gcd(abc/g^3) \equiv 0 \bmod 6, \text{ where } \gcd(a, b, c) = g.$$

(3) With the same n and R, for which initial polynomials $u_0, u_1, u_2 \in R_0 \times R_1 \times R_2$, is it true that

$$R/(u_a, u_b, u_c) \text{ is Artinian} \Leftrightarrow \begin{cases} \gcd(a, b, c) \equiv 0 \bmod 6, \\ \gcd(a+1, b+1, c+1) = 1, \\ \text{there exists } d \text{ such that } d+2 \not\equiv 0 \bmod t. \end{cases}$$

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