Set-theoretic reflection principles

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For a class $\mathcal{C}$ of structures (with a fixed notion $\mathcal{N}$ of substructure) and a property $\mathcal{P}$, the reflection cardinal of $(\mathcal{C}, \mathcal{P})$ is the minimal cardinal κ such that, for any $M \in \mathcal{C}$ of cardinality $> \kappa$, if M does not satisfies the property P, then there are stationarily many substructures N of M of cardinality $< \kappa$. If κ is the reflection cardinal of $(\mathcal{C}, \mathcal{N})$, we shall write $\kappa = \mathfrak{Refl}(\mathcal{C}, \mathcal{P})$.

By choosing $\mathcal{C}$, $\mathcal{N}$ and $\mathcal{P}$, we can represent many set-theoretic reflection statements. If, for example $\mathcal{P}$ is simply a contradiction, and $\mathcal{N}$ is the elementary submodel relation for some logic $\mathcal{L}$, then $\kappa = \mathfrak{Refl}(\mathcal{C}, \mathcal{P})$ is the strong form of Downward Löwenheim-Skolem Theorem down to $< \kappa$ for $\mathcal{L}$ (i.e. the assertion: for any structure $\mathfrak{A}$ of cardinality $\geq \kappa$ there are stationarily many $\mathcal{L}$-elementary substructures of $\mathfrak{A}$ of cardinality $< \kappa$).

Of these reflection statements, the cases $\aleph_2 = \mathfrak{Refl}(\mathcal{C}, \mathcal{P})$ and $2^{\aleph_0} = \mathfrak{Refl}(\mathcal{C}, \mathcal{P})$ seems to be of special interest. The former may be interpreted as a pronouncement that the first uncountable cardinal $\aleph_1$ captures the situation $\neg\mathcal{P}$ good enough while the latter as the pronouncement that the continuum is large enough in connection with the property $\mathcal{P}$.

For example, if we denote with $\mathcal{L}_{stat}$ the weak second-order logic (where the second order quantifiers run over countable infinite subsets of the underlying set of a structure) together with the stationarity quantifier (i.e. $stat\, X\, \varphi$ is to be interpreted as "there are stationarily many X such that φ"), if $\mathcal{C}^*$ denotes the class of all structures with the elementary submodel relation in $\mathcal{L}_{stat}$ as the notion of substructure, then the assertion $\aleph_2 = \mathfrak{Refl}(\mathcal{C}^*, \exists x\, (x \not\equiv x))$ is just the strong Löwenheim-Skolem theorem for the logic $\mathcal{L}_{stat}$ (i.e. the statement: for every uncountable structure $\mathfrak{A}$ there are stationarily many $\mathcal{L}_{stat}$-elementary substructures of $\mathfrak{A}$ of cardinality $\aleph_1$). It is easy to see that this reflection principle (even without the stationarity quantifier) implies the Continuum Hypothesis.

As the example above, stronger assertions among the reflection principles of the form $\aleph_2 = \mathfrak{Refl}(\mathcal{C}, \mathcal{P})$ imply the Continuum Hypothesis (see also [6]) while assertions of the form $2^{\aleph_0} = \mathfrak{Refl}(\mathcal{C}, \mathcal{P})$ tend to imply that the continuum is extremely large (see [1]).

Most of the natural assertions of the form $\aleph_2 = \mathfrak{Refl}(\mathcal{C}, \mathcal{P})$ or $2^{\aleph_0} = \mathfrak{Refl}(\mathcal{C}, \mathcal{P})$ involves some kind of countability in the property $\mathcal{P}$.

This is the case with the reflection assertion $\aleph_2 = \mathfrak{Refl}(\mathcal{C}_0, \mathcal{P}_0)$ where $\mathcal{C}_0$ is the class of all graphs with induced subgraphs as the notion of substructure and $\mathcal{P}_0$ is the property "of countable coloring number". It is shown that this assertion is equivalent to the Fodor-type Reflection Principle (FRP). Using a characterization of coloring number being less than a regular cardinal, it is easy to show that the strong Löwenheim-Skolem theorem $\aleph_2 = \mathfrak{Refl}(\mathcal{C}^*, \exists x\, (x \not\equiv x))$ above implies $\aleph_2 = \mathfrak{Refl}(\mathcal{C}_0, \mathcal{P}_0)$ (see also [5]).

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We can also consider the reflection number for the property obtained from these properties of countable character by replacing the countability by of cardinality κ. The reflection assertion for the property "of coloring number $< \kappa$" for regular $\kappa > \aleph_1$ is also equivalent to a higher cardinal version of FRP but it appeared that the straightforward generalization of the FRP does not work for this purpose ([3], [4]).

In this talk we will give a survey on recent results in connection with reflection statements including those mentioned above.

References

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