グラフのスペクトル解析における量子確率論の手法

Quantum Probabilistic Methods for Spectral Analysis of Graphs

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1. Introduction

A *graph* is a pair $G = (V, E)$, where V is a non-empty set of *vertices* and E a set of *edges*, namely, a set of unordered pairs of distinct vertices. Our main concern is spectral analysis of graphs, in particular, the asymptotic spectral distributions of the adjacency matrices of growing graphs. Here by growing graphs we mean a sequence of graphs $G_1 \subset G_2 \subset \cdots$ with the number of vertices going to the infinity. Typically, growing graphs are obtained from repeated application of a graph product or from an infinite family of parametrized graphs. The purpose of this note is to demonstrate how the ideas of quantum (noncommutative) probability theory is applied to such problems. In fact, we focus on (i) the method of quantum decomposition, and (ii) the use of various concepts of independence. Then asymptotic spectral distributions are derived along with quantum (noncommutative) version of the central limit theorem.

This note is based on the recent publication Obata [52], see also Hora–Obata [30]. The root of quantum probability traces back to von Neumann [58]. For relevant discussion and current trends see Accardi–Lu–Volovich [3], Meyer [41] and Parthasarathy [53]. For free probability see Nica–Speicher [44], Speicher [55] and Voiculescu [57].

2. Algebraic random variables and spectral distributions

Definition 2.1. Let $\mathcal{A}$ be a unital $*$-algebra over the complex number field $\mathbb{C}$ with the multiplication unit $1_{\mathcal{A}}$. A function $\varphi : \mathcal{A} \to \mathbb{C}$ is called a *state* on $\mathcal{A}$ if

(i) φ is linear; (ii) $\varphi(a^*a) \geq 0$; (iii) $\varphi(1_{\mathcal{A}}) = 1$.

The pair $(\mathcal{A}, \varphi)$ is called an *algebraic probability space*.

Definition 2.2. Let $(\mathcal{A}, \varphi)$ be an algebraic probability space. An element $a \in \mathcal{A}$ is called an *algebraic random variable* or a *random variable* for short. A random variable $a \in \mathcal{A}$ is called *real* if $a = a^*$.

For $a \in \mathcal{A}$ the quantity of the form: $\varphi(a^{\epsilon_1} a^{\epsilon_2} \cdots a^{\epsilon_m})$, $\epsilon_1, \epsilon_2, \ldots, \epsilon_m \in \{1, *\}$, is called a *mixed moment* of order m. For a real random variable $a = a^*$ the mixed moments are reduced to the *moment sequence:*

$$\varphi(a^m), \qquad m = 0, 1, 2, \ldots,$$

where $\varphi(a^m)$ is called the *mth moment* of a. By definition $\varphi(a^0) = 1$.

Definition 2.3. We say that two algebraic random variables a in $(\mathcal{A}, \varphi)$ and b in $(\mathcal{B}, \psi)$ are *moment equivalent* and write

$$a \overset{\mathrm{m}}{=} b$$

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if all of their mixed moments coincide. Two real random variables $a = a^*$ and $b = b^*$ are *moment equivalent*, i.e., $a \overset{\mathrm{m}}{=} b$ if their moment sequences coincide:

$$\varphi(a^m) = \psi(b^m) \quad \text{for all } m \geq 0.$$

Let $\mathfrak{P}_{\mathrm{fm}}(\mathbb{R})$ denote the set of all probability measures having finite moments of all orders. The *mth moment* of $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbb{R})$ is defined by

$$M_m(\mu) = \int_{\mathbb{R}} x^m \mu(dx), \qquad m \geq 0.$$

By definition $M_0(\mu) = 1$.

Theorem 2.4. *Let $(\mathcal{A}, \varphi)$ be an algebraic probability space. For a real random variable $a = a^* \in \mathcal{A}$ there exists a probability measure $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbb{R})$ such that*

$$\varphi(a^m) = \int_{-\infty}^{+\infty} x^m \mu(dx) = M_m(\mu), \qquad m \geq 0. \tag{2.1}$$

In fact, the existence of such a probability measure μ follows from the Hamburger theorem [19]. Note that μ is not uniquely determined in general. It is uniquely determined if and only if $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbb{R})$ is the solution to a determinate moment problem.

Definition 2.5. A probability measure μ satisfying (2.1) is called the *spectral distribution* of a in φ.

3. Interacting Fock spaces and quantum decomposition

Definition 3.1. A pair of finite or infinite sequences $(\{\omega_n\}, \{\alpha_n\})$ are called *Jacobi coefficients* if one of the following conditions is fulfilled:

(i) [infinite type] both are infinite sequences such that $\omega_n > 0$ and $\alpha_n \in \mathbb{R}$ for all $n \geq 1$,

(ii) [finite type] both are finite sequences of the form $(\{\omega_n\}_{n=1}^{d-1}, \{\alpha_n\}_{n=1}^{d})$, where $\omega_n > 0$, $\alpha_n \in \mathbb{R}$ and $d \geq 1$. (For $d = 1$ we understand that $\{\omega_n\}$ is an empty sequence.)

Let $\mathfrak{J}$ denote the set of such Jacobi coefficients.

With Jacobi coefficients $(\{\omega_n\}, \{\alpha_n\}) \in \mathfrak{J}$ we associate a Hilbert space Γ with complete orthonormal basis $\{\Phi_n\}$ and three linear operators B^+, B^-, B° defined by

$$B^+\Phi_n = \sqrt{\omega_{n+1}}\,\Phi_{n+1}, \quad n \geq 0, \tag{3.1}$$

$$B^-\Phi_0 = 0; \quad B^-\Phi_n = \sqrt{\omega_n}\,\Phi_{n-1}, \quad n \geq 1, \tag{3.2}$$

$$B^\circ\Phi_n = \alpha_{n+1}\Phi_n, \quad n \geq 0. \tag{3.3}$$

These actions are illustrated in Figure 1. If the Jacobi coefficients are of infinite type, we have $\dim\Gamma = \infty$. If it is of finite type, say $(\{\omega_n\}_{n=1}^{d-1}, \{\alpha_n\}_{n=1}^{d})$, we have $\dim\Gamma = d$ and set $B^+\Phi_{d-1} = 0$.

Definition 3.2. The quintuple $(\Gamma, \{\Phi_n\}, B^+, B^-, B^\circ)$ obtained as above is called an *interaction Fock space* associated with Jacobi coefficients $(\{\omega_n\}, \{\alpha_n\}) \in \mathfrak{J}$. The operators B^+, B^- and B° are called the *creation, annihilation* and *conservation operators*, respectively. When $\alpha_n \equiv 0$ we omit to refer to $B^\circ = 0$.

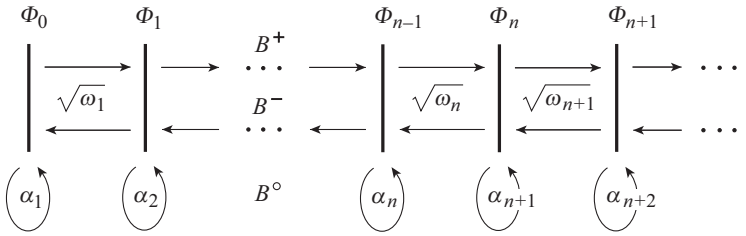

Figure 1: The actions of B^+, B^-, B°

Let Γ_0 be a linear space spanned by $\{\Phi_n\}$ and $\mathcal{L}(\Gamma_0)$ denote the space of all linear operators on Γ_0 whose adjoint operators, restricted to Γ_0, are linear operators on Γ_0. Then $\mathcal{L}(\Gamma_0)$ becomes a unital $*$-algebra in a natural manner. By definition $B^+, B^-, B^\circ \in \mathcal{L}(\Gamma_0)$ and we have

$$(B^+)^* = B^-, \qquad (B^-)^* = B^+, \qquad (B^\circ)^* = B^\circ.$$

Since $B^-B^+\Phi_n = \omega_{n+1}\Phi_n$ and $B^+B^-\Phi_n = \omega_n\Phi_n$ for $n \geq 0$, where we understand that $\omega_0 = 0$, it is convenient to write

$$B^-B^+ = \omega_{N+1}\,, \qquad B^+B^- = \omega_N\,, \qquad [B^-, B^+] = \omega_{N+1} - \omega_N\,,$$

where N is the number operator defined by $N\Phi_n = n\Phi_n$.

Equipped with the vector state Φ_0, called the *vacuum state*, $(\mathcal{L}(\Gamma_0), \Phi_0)$ becomes an algebraic probability space, where B^+, B^- and B° are regarded as random variables. In particular, the real random variable $B^+ + B^- + B^\circ$ is important. Recall that the spectral distribution is a probability measure μ satisfying

$$\langle \Phi_0, (B^+ + B^- + B^\circ)^m \Phi_0 \rangle = \int_{\mathbb{R}} x^m \mu(dx), \qquad m \geq 0. \tag{3.4}$$

Theorem 3.3. *Let $(\Gamma, \{\Phi_n\}, B^+, B^-, B^\circ)$ be the interacting Fock space associated with $(\{\omega_n\}, \{\alpha_n\}) \in \mathfrak{J}$. Then the vacuum spectral distribution of $B^+ + B^- + B^\circ$ coincides with the orthogonalizing probability measure for the polynomials $\{P_n(x)\}$ defined by*

$$P_0(x) = 1, \qquad P_1(x) = x - \alpha_1, \tag{3.5}$$

$$xP_n(x) = P_{n+1}(x) + \alpha_{n+1}P_n(x) + \omega_n P_{n-1}(x), \quad n \geq 1. \tag{3.6}$$

The essential step is to show that the linear operator $B^+ + B^- + B^\circ$ intertwines with the multiplication operator by x through the isometry $U : \Gamma_0 \to L^2(\mathbb{R}, \mu)$ determined by the correspondence $\Phi_n \leftrightarrow P_n(x)/\|P_n\|$. The proof requires some basic properties of the orthogonal polynomials as well.

Theorem 3.4. *Let $a = a^*$ be a real random variable in an algebraic probability space $(\mathcal{A}, \varphi)$. Then there exist unique Jacobi coefficients $(\{\omega_n\}, \{\alpha_n\}) \in \mathfrak{J}$ such that, letting $(\Gamma, \{\Phi_n\}, B^+, B^-, B^\circ)$ be the associated interacting Fock space, we have*

$$\varphi(a^m) = \langle \Phi_0, (B^+ + B^\circ + B^-)^m \Phi_0 \rangle, \qquad m \geq 0. \tag{3.7}$$

From (3.7) we obtain

$$a \overset{\mathrm{m}}{=} B^+ + B^\circ + B^-, \tag{3.8}$$

which is called the *quantum decomposition* of a random variable $a = a^*$.

Example 3.5 (Boson Fock space)**.** This is the interacting Fock space associated with Jacobi coefficients $(\{\omega_n = n\}, \{\alpha_n \equiv 0\})$. The *canonical commutation relation (CCR)* holds:

$$[B^-, B^+] = B^-B^+ - B^+B^- = I. \tag{3.9}$$

The vacuum spectral distribution of $B^+ + B^-$ is the standard normal (Gaussian) distribution, i.e.,

$$\langle \Phi_0, (B^+ + B^-)^m \Phi_0 \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^m e^{-x^2/2} dx, \qquad m \geq 0. \tag{3.10}$$

Example 3.6 (Free Fock space)**.** This is the interacting Fock space associated with Jacobi coefficients $(\{\omega_n \equiv 1\}, \{\alpha_n \equiv 0\})$. The *free commutation relation* holds:

$$B^-B^+ = I. \tag{3.11}$$

The vacuum spectral distribution of $B^+ + B^-$ is the normalized Wigner semicircle law, i.e.,

$$\langle \Phi_0, (B^+ + B^-)^m \Phi_0 \rangle = \frac{1}{2\pi} \int_{-2}^{+2} x^m \sqrt{4 - x^2}\, dx, \qquad m \geq 0. \tag{3.12}$$

Example 3.7 (Fermion Fock space)**.** This is the interacting Fock space associated with Jacobi coefficients $(\{\omega_1 = 1\}, \{\alpha_1 = \alpha_2 = 0\})$. The *anti-commutation relation* holds:

$$B^-B^+ + B^+B^- = I. \tag{3.13}$$

Identifying Γ_0 with $\mathbb{C}^2$, we obtain matrix notation:

$$\Phi_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Phi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{3.14}$$

The vacuum spectral distribution of $B^+ + B^-$ is the Bernoulli distribution with mean 0 and variance 1, i.e.,

$$\langle \Phi_0, (B^+ + B^-)^m \Phi_0 \rangle = \int_{\mathbb{R}} x^m \mu_{\mathrm{B}}(dx), \quad \mu_{\mathrm{B}} = \frac{1}{2}(\delta_{+1} + \delta_{-1}), \quad m \geq 0. \tag{3.15}$$

4. Quantum decomposition of classical random variables

Let $(\Omega, \mathcal{F}, P)$ be a (classical) probability space and consider the set of all $\mathbb{C}$-valued random variables having finite moments of all orders, i.e.,

$$L^{\infty-}(\Omega, \mathcal{F}, P) = \bigcap_{1 \leq p < \infty} L^p(\Omega, \mathcal{F}, P)$$

Equipped with the usual pointwise operations, $L^{\infty-}(\Omega, \mathcal{F}, P)$ becomes a unital $*$-algebra. Moreover, equipped with the expectation

$$\mathbf{E}[X] = \int_\Omega X(\omega) P(d\omega), \qquad X \in L^{\infty-}(\Omega, \mathcal{F}, P),$$

$(L^{\infty-}(\Omega, \mathcal{F}, P), \mathbf{E})$ becomes an algebraic probability space.

Let X be a $\mathbb{R}$-valued random variable in $L^{\infty-}(\Omega, \mathcal{F}, P)$ and μ_X the probability distribution of X. From the obvious relation

$$\mathbf{E}[X^m] = \int_\Omega X(\omega)^m P(d\omega) = \int_{\mathbb{R}} x^m \mu_X(dx) = M_m(\mu_X)$$

we see that the spectral distribution of X in the sense of the algebraic random variable is nothing else but the probability distribution of X.

Example 4.1. The coin toss is modelled by a classical random variable X defined by $P(X = 1) = P(X = -1) = 1/2$. It follows from Example 3.7 that

$$X \overset{\mathrm{m}}{=} B^+ + B^-, \tag{4.1}$$

where B^+ and B^- are the Fermion creation and annihilation operators. Moreover, (4.1) gives rise to the quantum decomposition of X.

Example 4.2. Let X be a classical random variable obeying the standard normal distribution. We then see from Example 3.5 that

$$X \overset{\mathrm{m}}{=} B^+ + B^-, \tag{4.2}$$

where B^+ and B^- are the Boson creation and annihilation operators. Moreover, (4.2) gives rise to the quantum decomposition of X.

Example 4.3. Let $(\Gamma, \{\Phi_n\}, B^+, B^-)$ be the Boson Fock space. Then the vacuum distribution of $(B^+ + \sqrt{\lambda})(B^- + \sqrt{\lambda})$ coincides with the Poisson distribution with parameter $\lambda > 0$. Since

$$(B^+ + \sqrt{\lambda})(B^- + \sqrt{\lambda}) = N + \sqrt{\lambda}(B^+ + B^-) + \lambda,$$

where $N = B^+B^-$ is the number operator, the Jacobi coefficients of the Poisson distribution with parameter λ are given by

$$\omega_n = \lambda n, \qquad \alpha_n = n - 1 + \lambda, \qquad n \geq 1.$$

Example 4.4. A parallel argument is applied to the free Fock space $(\Gamma, \{\Phi_n\}, B^+, B^-)$. For $\lambda > 0$ the vacuum spectral distribution of $(B^+ + \sqrt{\lambda})(B^- + \sqrt{\lambda})$ is a unique probability measure determined by Jacobi coefficients $(\{\omega_n \equiv \lambda\}, \{\alpha_1 = \lambda, \alpha_2 = \alpha_3 = \cdots = \lambda + 1\})$. It is called the *free Poisson distribution* or the *Marchenko–Pastur distribution* with parameter λ. In fact, the free Poisson distribution is obtained from the free Meixner distribution by affine transformation. For more details see e.g., Hiai–Petz [24], Hora–Obata [30], Nica–Speicher [44].

In general, a classical random variable having finite moments of all orders admits the quantum decomposition:

$$X \overset{\mathrm{m}}{=} B^+ + B^\circ + B^-. \tag{4.3}$$

In fact, the Jacobi coefficients $(\{\omega_n\}, \{\alpha_n\})$ are first obtained from the probability distribution of X using the three-term recurrence relation. Then we take the associated interacting Fock space which gives rise to (4.3). The quantum decomposition makes us possible to study "non-commutative structure" of a random variable $a = a^*$, in particular, of a classical random variable X.

The commutators $[B^-, B^+], [B^\circ, B^+]$ and $[B^\circ, B^-]$ are determined by the Jacobi coefficients $(\{\omega_n\}, \{\alpha_n\})$. Since a probability measure $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbb{R})$ is determined by the Jacobi coefficients $(\{\omega_n\}, \{\alpha_n\})$ (in the sense of moments), so is by the commutators $[B^-, B^+], [B^\circ, B^+], [B^\circ, B^-]$. There are some attempts to classify probability measures in terms of (Lie) algebraic structure of interacting Fock spaces.

5. Spectral Distributions of Graphs

Definition 5.1. A *graph* is a pair $G = (V, E)$, where V is a non-empty set of *vertices* and E a set of *edges*, namely, a set of unordered pairs of distinct vertices. We say that two vertices $x, y \in V$ are *adjacent* if $\{x, y\} \in E$. In this case we also write $x \sim y$. A graph $G = (V, E)$ is called *finite* if V is a finite set.

The *degree* of a vertex $x \in V$ is defined by

$$\deg(x) = \deg_G(x) = |\{y \in V\,;\, y \sim x\}|.$$

A graph G is called *locally finite* if $\deg(x) < \infty$ for all $x \in V$. In this note a graph is always assumed to be locally finite.

Definition 5.2. The *adjacency matrix* of a graph $G = (V, E)$ is a matrix $A = A[G]$ with index set $V \times V$ defined by

$$(A)_{xy} = \begin{cases} 1, & \text{if } x \sim y, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{A}(G)$ be the set of all polynomials in A with complex coefficients, which becomes a unital $*$-algebra in a natural manner and is called the *adjacency algebra* of G.

Once a state φ is given on $\mathcal{A}(G)$, we regard the adjacency matrix $A = A^*$ as a real random variable of the algebraic probability space $(\mathcal{A}(G), \varphi)$. Then by Theorem 2.4 there exists a probability measure $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbb{R})$ such that

$$\varphi(A^m) = M_m(\mu) = \int_{\mathbb{R}} x^m \mu(dx), \qquad m \geq 0.$$

We call μ the *spectral distribution* of A in the state φ.

Normalized trace. Let $G = (V, E)$ be a finite graph. The normalized trace is defined by

$$\varphi_{\mathrm{tr}}(a) = \frac{1}{|V|} \operatorname{Tr} a = \frac{1}{|V|} \sum_{x \in V} (a)_{xx} = \frac{1}{|V|} \sum_{x \in V} \langle e_x, a e_x \rangle, \qquad a \in \mathcal{A}(G),$$

where $\{e_x\,;\, x \in V\}$ is the canonical basis. The spectral distribution of A in φ_{tr} coincides with the eigenvalue distribution of G, which is defined by

$$\mu = \frac{1}{|V|} \sum_{i=1}^{s} m_i \delta_{\lambda_i}\,, \qquad \operatorname{Spec}(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_s \\ m_1 & m_2 & \dots & m_s \end{pmatrix}.$$

Vacuum state at a vertex. Suppose we are given a distinguished vertex $o \in V$. The vacuum state at o is defined by

$$\langle a \rangle_o = \langle e_o, a e_o \rangle = (a)_{oo}\,, \qquad a \in \mathcal{A}(G).$$

Let μ be the spectral distribution of the adjacency matrix A in the vacuum state $\langle \cdot \rangle_o$. Then we have

$$M_m(\mu) = \int_{\mathbb{R}} x^m \mu(dx) = \langle A^m \rangle_o = W_m(o, o; G), \qquad m \geq 0, \tag{5.1}$$

where $W_m(x, y; G) = (A^m)_{xy} = \langle e_x, A^m e_y \rangle$ denotes the number of m-step walks connecting x and y.

6. Fock spaces associated with graphs

Let $G = (V, E)$ be a connected graph with a distinguished vertex $o \in V$. We set

$$V_n = \{x \in V\,;\, \partial(x, o) = n\}, \qquad n \geq 0. \tag{6.1}$$

Obviously, $V_0 = \{o\}$, $V_1 = \{x \in V\,;\, x \sim o\}$, and $V_m \cap V_n = \emptyset$ for $m \neq n$. Thus, we obtain a partition of V:

$$V = \bigcup_n V_n\,, \tag{6.2}$$

which is called the *stratification* of G with respect to $o \in V$.

Associated to (6.1) we define $\Phi_n \in C_0(V)$ by

$$\Phi_n = \frac{1}{\sqrt{|V_n|}} \sum_{x \in V_n} e_x\,, \qquad n \geq 0.$$

Here we note that $|V_n| < \infty$ by local finiteness. By definition, $\Phi_0 = e_o$ and $\langle \Phi_m, \Phi_n \rangle = \delta_{mn}$. Let Γ_0 denote the subspace of $C_0(V)$ spanned by $\{\Phi_n\}$. The Hilbert space obtained by completing Γ_0 is called the *Fock space* associated with a rooted graph $(G; o)$ and is denoted by $\Gamma(G; o)$. By construction, $\{\Phi_n\}$ form an orthonormal basis of $\Gamma(G; o)$.

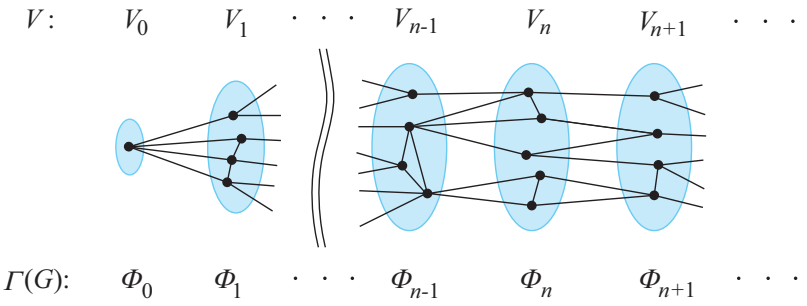

Figure 2: Stratification and Fock space

For $\epsilon \in \{+, -, \circ\}$ and $x \in V$ we set

$$\omega_\epsilon(x) = \{y \in V\,;\, y \sim x,\ \partial(o, y) = \partial(o, x) + \epsilon\}, \tag{6.3}$$

where the symbols $+, -, \circ$ correspond respectively to the numbers $+1, -1, 0$. We define three matrices A^+, A°, A^- by

$$(A^\epsilon)_{yx} = \begin{cases} 1, & \text{if } y \in \omega_\epsilon(x), \\ 0, & \text{otherwise,} \end{cases} \qquad \epsilon \in \{+, -, \circ\},$$

or equivalently,

$$A^\epsilon e_x = \sum_{y \in \omega_\epsilon(x)} e_y\,, \qquad x \in V. \tag{6.4}$$

Then we have the *quantum decomposition* of the adjacency matrix:

$$A = A^+ + A^- + A^\circ. \tag{6.5}$$

Note that

$$(A^+)^* = A^-, \qquad (A^-)^* = A^+, \qquad (A^\circ)^* = A^\circ. \tag{6.6}$$

Theorem 6.1. *Let G be a connected graph with a distinguished vertex $o \in V$. Let $\Gamma(G;o)$ be the associated Fock space and $A = A^+ + A^- + A^\circ$ the quantum decomposition of the adjacency matrix A. If $\Gamma(G;o)$ is invariant under the actions of A^ϵ, then $(\Gamma(G;o), \{\Phi_n\}, A^+, A^-, A^\circ)$ is an interacting Fock space associated with Jacobi coefficients $(\{\omega_n\}, \{\alpha_n\})$ defined by*

$$\omega_n = \frac{|V_n|}{|V_{n-1}|}\,|\omega_-(x)|^2, \quad x \in V_n\,; \quad \alpha_n = |\omega_\circ(x)|, \quad x \in V_{n-1}\,, \tag{6.7}$$

for $n \geq 1$, where the right-hand sides are independent of the choice of x.

Corollary 6.2. *Notations and assumptions being the same as in Theorem 6.1, the spectral distribution of A in the vacuum state at $o \in V$ is a probability measure with Jacobi coefficients $(\{\omega_n\}, \{\alpha_n\})$ defined in* (6.7).

7. Distance-Regular Graphs

There is an interesting family of graphs for which $\Gamma(G;o)$ is invariant under the actions of A^ϵ. In fact, distance-regular graphs are closely related to orthogonal polynomials.

Definition 7.1. A connected graph $G = (V, E)$ is called *distance-regular* if for any $i, j, k \in \{0, 1, 2, \dots\}$, the number

$$p_{ij}^k = |\{z \in V\,;\, \partial(z,x) = i,\, \partial(y,z) = j\}|$$

is independent of the choice of $x, y \in V$ with $\partial(x,y) = k$. The numbers $\{p_{ij}^k\}$ are called the *intersection numbers* of a distance-regular graph G.

Definition 7.2. Picking up a subset of $\{p_{ij}^k\}$, we define

$$a_n = p_{1n}^n\,, \qquad b_n = p_{1\,n+1}^n \qquad c_n = p_{1\,n-1}^n\,, \qquad n \geq 0, \tag{7.1}$$

where $c_0 = 0$ by definition. For a finite distance-regular graph G with $d = \mathrm{diam}\,(G)$, the array

$$\begin{pmatrix} \{c_n\} \\ \{a_n\} \\ \{b_n\} \end{pmatrix} = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_d \\ a_0 & a_1 & a_2 & \cdots & a_d \\ b_0 & b_1 & b_2 & \cdots & b_d \end{pmatrix}$$

is called the *intersection array* of G. If G is infinite, the array becomes infinite.

A distance-regular graph is regular with degree $b_0 = p_{11}^0$. Therefore, $a_n + b_n + c_n = b_0$ for all $n \geq 0$. In particular, $a_0 = c_0 = 0$. Since $a_n + b_n + c_n = b_0$ is constant, the row of $a_0, a_1, \dots$ in the intersection array is often omitted.

Theorem 7.3. *Let G be a distance-regular graph and $(\{c_n\}, \{a_n\}, \{b_n\})$ the intersection array. Choosing a root $o \in V$, let $\Gamma = \Gamma(G, o)$ be the associated Fock space with basis $\{\Phi_0 = e_o, \Phi_1, \Phi_2, \dots\}$ and $A = A^+ + A^- + A^\circ$ the quantum decomposition of the adjacency matrix. Then $(\Gamma, \{\Phi_n\}, A^+, A^-, A^\circ)$ is an interacting Fock space associated with Jacobi coefficients $(\{\omega_n\}, \{\alpha_n\})$ given by*

$$\omega_n = b_{n-1}c_n\,, \qquad \alpha_n = a_{n-1}, \qquad n \geq 1. \tag{7.2}$$

In particular, the above Jacobi coefficients are independent of the choice of $o \in V$.

It is easily verified that the normalized trace φ_{tr} and the vacuum state at any vertex coincide on the adjacency algebra $\mathcal{A}(G)$ of a distance-regular graph.

8. Asymptotic spectral distributions

8.1. Growing distance-regular graphs in general

Let $G^{(\nu)}$ be growing distance-regular graphs and $(\{c_n(\nu)\}, \{a_n(\nu)\}, \{b_n(\nu)\})$ their intersection arrays. For each ν we have the quantum decomposition of the adjacency matrix

$$A_\nu = A_\nu^+ + A_\nu^- + A_\nu^\circ$$

and we have the interacting Fock space $(\Gamma_\nu, \{\Phi_n^{(\nu)}\}, A_\nu^+, A_\nu^-, A_\nu^\circ)$ associated with Jacobi coefficients

$$\omega_n(\nu) = b_{n-1}(\nu)c_n(\nu)\,, \qquad \alpha_n(\nu) = a_{n-1}(\nu), \qquad n \geq 1. \tag{8.1}$$

In general, $\omega_n(\nu)$ diverges as $\nu \to \infty$. But, taking a suitable scaling limit, we may hope to get reasonable limits $(\{\omega_n\}, \{\alpha_n\})$ from which we obtain the asymptotic spectral distribution of the growing graphs. This program can be performed. Before mentioning the main result, we need to prepare a notion of convergence.

Definition 8.1. For each $n \geq 1$ let $(\mathcal{A}_n, \varphi_n)$ be an algebraic probability space and $(a_{n,1}, \ldots, a_{n,d})$ be a d-dimensional random vector, i.e., $a_{n,i} \in \mathcal{A}_n$ for $1 \leq i \leq d$. Let $(\mathcal{B}, \psi)$ be another algebraic probability space and $(b_1, \ldots, b_d)$ a d-dimensional random vector. We say that $(a_{n,1}, \ldots, a_{n,d})$ *converges to* $(b_1, \ldots, b_d)$ *in moment* if

$$\lim_{n\to\infty} \varphi_n(a_{n,i_1}^{\epsilon_1} \cdots a_{n,i_m}^{\epsilon_m}) = \psi(b_{i_1}^{\epsilon_1} \cdots b_{i_m}^{\epsilon_m})$$

for any choice of $i_1, \ldots, i_m \in \{1, \ldots, d\}$, $\epsilon_1, \ldots, \epsilon_m \in \{1, *\}$ and $m \geq 1$. In that case we write

$$(a_{n,1}, \ldots, a_{n,d}) \xrightarrow{\text{m}} (b_1, \ldots, b_d).$$

In particular, for real random variables $a_n = a_n^* \in \mathcal{A}_n$, $n \geq 1$, and $b = b^* \in \mathcal{B}$, we say that a_n *converges to* b *in moment* if

$$\lim_{n\to\infty} \varphi_n(a_n^m) = \psi(b^m)$$

for all $m \geq 0$. In that case we write $a_n \xrightarrow{\text{m}} b$.

Theorem 8.2 (QCLT for growing distance-regular graphs)**.** *Notations and assumptions being as above, the degree is denoted by* $\kappa(\nu) = b_0(\nu)$. *Assume that*

$$\omega_n = \lim_{\nu\to\infty} \frac{b_{n-1}(\nu)c_n(\nu)}{\kappa(\nu)}\,, \qquad \alpha_n = \lim_{\nu\to\infty} \frac{a_{n-1}(\nu)}{\sqrt{\kappa(\nu)}} \tag{8.2}$$

exist for all $n \geq 1$ *and* $(\{\omega_n\}, \{\alpha_n\})$ *are Jacobi coefficients. Let* $(\Gamma, \{\Psi_n\}, B^+, B^-, B^\circ)$ *be the associated interacting Fock space. Then we have*

$$\left(\frac{A_\nu^+}{\sqrt{\kappa(\nu)}}, \frac{A_\nu^-}{\sqrt{\kappa(\nu)}}, \frac{A_\nu^\circ}{\sqrt{\kappa(\nu)}}\right) \xrightarrow{\text{m}} (B^+, B^-, B^\circ), \qquad \text{as } \nu \to \infty. \tag{8.3}$$

After the normalization as in (8.2) we obtain $\omega_1 = 1$ and $\alpha_1 = 0$. That is, the limit distribution has mean 0 and variance 1.

8.2. Hamming graphs

Let $N \geq 1$ and $d \geq 1$. Set $F = \{1, 2, \ldots, N\}$ and consider the d fold Cartesian product of F:

$$F^d = \{x = (\xi_1, \ldots, \xi_d)\,;\, \xi_i \in F,\ 1 \leq i \leq d\}.$$

For $x = (\xi_1, \ldots, \xi_d)$ and $y = (\eta_1, \ldots, \eta_d)$ define

$$\partial(x, y) = |\{1 \leq i \leq d\,;\, \xi_i \neq \eta_i\}|.$$

Then ∂ becomes a metric on F^d, which is called the *Hamming distance*. A graph on $V = F^d$ with edge set $E = \{\{x, y\}\,;\, x, y \in V,\, \partial(x, y) = 1\}$ is called a *Hamming graph* and is denoted by $H(d, N)$.

The Hamming graph $H(d, N)$ is distance-regular and the intersection array is given by

$$a_n = n(N-2), \quad b_n = (d-n)(N-1), \quad c_n = n, \tag{8.4}$$

for $0 \leq n \leq d = \mathrm{diam} H(d, N)$. In particular, the degree is given by

$$\kappa_{d,N} = d(N-1).$$

The Jacobi coefficients of the vacuum spectral distribution of A of $H(d, N)$ are given by

$$\begin{aligned} \omega_n &= n(d-n+1)(N-1), \qquad 1 \leq n \leq d, \\ \alpha_n &= (n-1)(N-2), \qquad 1 \leq n \leq d+1. \end{aligned}$$

These diverges as $d \to \infty$ and $N \to \infty$. We see from

$$\langle e_o, Ae_o \rangle = 0, \qquad \langle e_o, A^2 e_o \rangle = \deg(o) = \kappa_{d,N} = d(N-1),$$

that a reasonable normalization is given by

$$\frac{A}{\sqrt{d(N-1)}} = \frac{A^+}{\sqrt{d(N-1)}} + \frac{A^-}{\sqrt{d(N-1)}} + \frac{A^\circ}{\sqrt{d(N-1)}}$$

and we have

$$\frac{A^+}{\sqrt{d(N-1)}}\,\Phi_n = \sqrt{(n+1)\left(1 - \frac{n}{d}\right)}\,\Phi_{n+1}\,, \tag{8.5}$$

$$\frac{A^-}{\sqrt{d(N-1)}}\,\Phi_n = \sqrt{n\left(1 - \frac{n-1}{d}\right)}\,\Phi_{n-1}\,, \tag{8.6}$$

$$\frac{A^\circ}{\sqrt{d(N-1)}}\,\Phi_n = n\sqrt{\frac{N-2}{d}}\sqrt{\frac{N-2}{N-1}}\,\Phi_n\,. \tag{8.7}$$

The coefficients in the right-hand sides converge as $d \to \infty$ and $N \to \infty$ with the scaling balance condition:

$$\frac{N}{d} \to \tau \geq 0. \tag{8.8}$$

The limit actions are described by the Boson Fock space $(\Gamma, \{\Psi_n\}, B^+, B^-)$ associated with the Jacobi coefficients $(\{\omega_n = n\}, \{\alpha_n \equiv 0\})$. In fact, we have formally obtain:

$$\lim_{\substack{N/d \to \tau \\ d,N \to \infty}} \frac{A^\epsilon}{\sqrt{d(N-1)}} = B^\epsilon, \qquad \epsilon \in \{+, -, \circ\}, \tag{8.9}$$

where $B^\circ = \sqrt{\tau}\, B^+ B^-$. We claim (8.9) in the sense of moment convergence.

Theorem 8.3 (quantum CLT for Hamming graphs)**.** *Let $A_{d,N}$ be the adjacency matrix of the Hamming graph $H(d,N)$, and $A_{d,N} = A^+_{d,N} + A^-_{d,N} + A^\circ_{d,N}$ the quantum decomposition with respect to an arbitrary chosen root $o = o_{d,N}$. Let $(\Gamma, \{\Psi_n\}, B^+, B^-)$ be the Boson Fock space. Then we have*

$$\left(\frac{A^+_{d,N}}{\sqrt{d(N-1)}}, \frac{A^-_{d,N}}{\sqrt{d(N-1)}}, \frac{A^\circ_{d,N}}{\sqrt{d(N-1)}} \right) \xrightarrow{\mathrm{m}} (B^+, B^-, \sqrt{\tau}\, B^+B^-), \tag{8.10}$$

as $N \to \infty$ and $d \to \infty$ with $N/d \to \tau \geq 0$. In particular, for $m \geq 0$ we have

$$\lim_{\substack{N/d \to \tau \\ d,N \to \infty}} \left\langle e_o, \left(\frac{A_{d,N}}{\sqrt{d(N-1)}} \right)^m e_o \right\rangle = \langle \Psi_0, (B^+ + B^- + \sqrt{\tau}\, B^+B^-)^m \Psi_0 \rangle. \tag{8.11}$$

Suppose $\tau = 0$. Then $B^+ + B^- + \sqrt{\tau}\, B^+B^- = B^+ + B^-+$ is a sum of annihilation and creation operators of Boson Fock space so the vacuum spectral distribution is the standard normal distribution, see Section 3. Since the normal distribution is the solution to a determinate moment problem, the normalized spectral distribution of the Hamming graph $H(d,N)$ converges to the standard normal distribution weakly too.

Suppose $\tau > 0$. We have

$$B^+ + B^- + \sqrt{\tau}\, B^+B^- = \sqrt{\tau} \left(B^+ + \frac{1}{\sqrt{\tau}} \right) \left(B^- + \frac{1}{\sqrt{\tau}} \right) - \frac{1}{\sqrt{\tau}}. \tag{8.12}$$

It is known (Section 3) that the vacuum distribution of $(B^+ + 1/\sqrt{\tau})(B^- + 1/\sqrt{\tau})$ is the Poisson distribution with parameter $1/\tau$. Thus, the vacuum spectral distribution of (8.12) is obtained from the Poisson distribution by affine transformation (dilation and translation). Of course, it is described explicitly though omitted.

8.3. Homogeneous trees and Kesten distributions

For $\kappa \geq 2$ let T_κ denote the *homogeneous tree* of degree $\kappa \geq 2$, i.e., a unique κ-regular connected graph without cycles. Of course, T_κ is an infinite graph. As is easily verified, T_κ is a distance-regular graph with intersection array:

$$\begin{pmatrix} \{c_n\} \\ \{a_n\} \\ \{b_n\} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & \dots \\ 0 & 0 & 0 & \dots \\ \kappa & \kappa-1 & \kappa-1 & \dots \end{pmatrix}$$

Hence, the spectral distribution μ_κ in the vacuum state at an arbitrary chosen root $o \in V$ is a probability measure corresponding to the Jacobi coefficients given by

$$\omega_1 = \kappa, \quad \omega_2 = \omega_3 = \cdots = \kappa - 1; \qquad \alpha_n = 0, \quad n \geq 1. \tag{8.13}$$

Theorem 8.4 (quantum CLT for homogeneous trees)**.** *For $\kappa \geq 2$ let A_κ be the adjacency matrix of the homogeneous tree T_κ and $A_\kappa = A^+_\kappa + A^-_\kappa$ the quantum decomposition with respect to an arbitrary chosen root $o = o_\kappa$. Let $(\Gamma, \{\Psi_n\}, B^+, B^-)$ be the free Fock space associated with the Jacobi coefficients $(\{\omega_n \equiv 1\}, \{\alpha_n \equiv 0\})$. Then, we have*

$$\left(\frac{A^+_\kappa}{\sqrt{\kappa}}, \frac{A^-_\kappa}{\sqrt{\kappa}} \right) \xrightarrow{\mathrm{m}} (B^+, B^-), \quad \kappa \to \infty.$$

Corollary 8.5. *We have*

$$\lim_{\kappa\to\infty}\left\langle e_o, \left(\frac{A_\kappa}{\sqrt{\kappa}}\right)^m e_o\right\rangle = \langle\Psi_0, (B^+ + B^-)^m\Psi_0\rangle, \qquad m \geq 0. \tag{8.14}$$

Therefore, the normalized spectral distribution of the adjacency matrix of T_κ converges to the normalized Wigner semicircle law in the sense of moments and weak convergence as $\kappa \to \infty$.

In fact, from (8.13) we may easily obtain the vacuum spectral distribution of T_κ, which is given by $\mu_\kappa(dx) = \rho_\kappa(x)dx$ with

$$\rho_\kappa(x) = \frac{\kappa}{2\pi}\,\frac{\sqrt{4(\kappa-1)-x^2}}{\kappa^2-x^2}\,, \qquad |x| \leq 2\sqrt{\kappa-1}\,. \tag{8.15}$$

This was derived first by Kesten [34]. The derivation is clarified by means of the quantum decomposition. It is also easy to verify the scaling limit of (8.15) converges to the Wigner semicircle law. This may be seen as a prototype of free central limit theorem.

8.4. Spidernets and free Meixner distributions

Definition 8.6. Let a, b, c be integers such that $a \geq 1$, $b \geq 2$ and $1 \leq c \leq b-1$. A *spidernet* is a graph $G = (V, E)$ with a root $o \in V$ such that

$$\begin{cases} \omega_+(o) = a, \\ \omega_-(o) = 0, \\ \omega_\circ(o) = 0, \end{cases} \qquad \begin{cases} \omega_+(x) = c, \\ \omega_-(x) = 1, \\ \omega_\circ(x) = b-1-c, \end{cases} \qquad \text{for } x \neq o, \tag{8.16}$$

where $\omega_\epsilon(x) = |\{y \in V\,;\, y \sim x,\ \partial(o,y) = \partial(o,x) + \epsilon\}|$ for $x \in V$ as usual. Such a spidernet is denoted by $S(a, b, c)$.

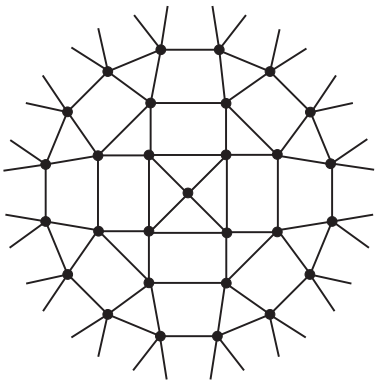

Figure 3: Spidernet $S(4, 5, 2)$

In particular, $S(a, a, a-1) = T_a$ is a homogeneous tree of degree a. Note that a spidernet is not uniquely determined by the three parameters a, b, c in general. A spidernet is not distance-regular in general. Nevertheless, the method of quantum decomposition works well.

Theorem 8.7. *Let $G = S(a,b,c)$ be a spidernet with a root $o \in V$. Let A be the adjacency matrix and $A = A^+ + A^- + A^\circ$ the quantum decomposition. Then $(\Gamma(G), \{\Phi_n\}, A^+, A^-, A^\circ)$ becomes an interacting Fock space associated with the Jacobi coefficients*

$$\omega_1 = a,\ \omega_2 = \omega_3 = \cdots = c; \quad \alpha_1 = 0,\ \alpha_2 = \alpha_3 = \cdots = b - 1 - c. \tag{8.17}$$

Corollary 8.8. *The vacuum spectral distribution of the adjacency matrix A of a spidernet $G = S(a,b,c)$ at the root $o \in V$ is a unique probability measure corresponding to the Jacobi parameters given in* (8.17).

Definition 8.9. Let $p > 0$, $q \geq 0$ and $a \in \mathbb{R}$. A probability measure uniquely determined by the Jacobi coefficients

$$\{\omega_n\} = \{p, q, q, \dots\}, \qquad \{\alpha_n\} = \{0, a, a, \dots\},$$

is called the *free Meixner distribution* with parameters p, q, a. An explicit form is known, see e.g., Hora–Obata [30].

8.5. Asymptotic quantum decomposition

For the N-dimensional integer lattice $\mathbb{Z}^N$, $N \geq 2$, we cannot apply the method of quantum decomposition, but in the limit as $N \to \infty$ we obtain an interacting Fock space structure. This observation yields a general result.

Let G_ν be growing regular graph with distinguished vertices $o = o_\nu$. Each G_ν admits the stratification with respect to o_ν and we have the quantum decomposition of the adjacency matrix $A_\nu = A_\nu^+ + A_\nu^- + A_\nu^\circ$. In order to control the actions of quantum components we prepare three statistics for ω_ϵ. We define

$$\begin{aligned}
M(\omega_\epsilon|V_n) &= \frac{1}{|V_n|} \sum_{x \in V_n} |\omega_\epsilon(x)| \\
\Sigma^2(\omega_\epsilon|V_n) &= \frac{1}{|V_n|} \sum_{x \in V_n} \big\{ |\omega_\epsilon(x)| - M(\omega_\epsilon|V_n) \big\}^2 \\
L(\omega_\epsilon|V_n) &= \max\{ |\omega_\epsilon(x)| \,;\, x \in V_n \},
\end{aligned}$$

where the suffix ν is omitted. Each G_ν is assumed to be regular, of which the degree is denoted by $\kappa(\nu)$. We pose conditions for the asymptotics of the above statistics:

(A1) $\lim_{\nu \to \infty} \kappa(\nu) = \infty$;

(A2) For any $n \geq 1$,

$$\lim_{\nu \to \infty} M(\omega_-|V_n) \equiv \omega_n < \infty, \quad \lim_{\nu \to \infty} \Sigma^2(\omega_-|V_n) = 0, \quad \sup_\nu L(\omega_-|V_n) < \infty;$$

(A3) For any $n \geq 0$,

$$\lim_{\nu \to \infty} \frac{M(\omega_\circ|V_n)}{\sqrt{\kappa(\nu)}} \equiv \alpha_{n+1} < \infty, \quad \lim_{\nu \to \infty} \frac{\Sigma^2(\omega_\circ|V_n)}{\kappa(\nu)} = 0, \quad \sup_\nu \frac{L(\omega_\circ|V_n)}{\sqrt{\kappa(\nu)}} < \infty,$$

where V_n stands for the nth stratum of the stratification of G_ν.

Theorem 8.10 (quantum CLT for growing regular graphs)**.** *Let $G_\nu = (V_\nu, E_\nu)$ be growing regular graphs enjoying conditions (A1)–(A3). Let $(\Gamma, \{\Psi_n\}, B^+, B^-, B^\circ)$ be the interacting Fock space associated with the Jacobi coefficients $(\{\omega_n\}, \{\alpha_n\})$. Then, as random variables in the algebraic probability space $(\tilde{\mathcal{A}}(G_\nu), o_\nu)$, we have*

$$\left(\frac{A_\nu^+}{\sqrt{\kappa(\nu)}}, \frac{A_\nu^-}{\sqrt{\kappa(\nu)}}, \frac{A_\nu^\circ}{\sqrt{\kappa(\nu)}} \right) \xrightarrow{\mathrm{m}} (B^+, B^-, B^\circ), \quad \nu \to \infty.$$

Corollary 8.11. *Notations and assumptions being as in Theorem 8.10, we have*

$$\lim_{\nu\to\infty} \left\langle e_o, \left(\frac{A_\nu}{\sqrt{\kappa(\nu)}} \right)^m e_o \right\rangle = \langle \Psi_0, (B^+ + B^- + B^\circ)^m \Psi_0 \rangle, \qquad m \geq 0. \tag{8.18}$$

Therefore, the normalized spectral distribution of the adjacency matrix A_ν in the vacuum state at o_ν converges in moment to a probability measure associated with the Jacobi coefficients $(\{\omega_n\}, \{\alpha_n\})$.

The complete proof of the above statements is given by Hora–Obata [31]. Conditions (A1)–(A3) say that the actions of the quantum components A_ν^ϵ coincide asymptotically with those of B^ϵ in the interacting Fock space associated with Jacobi coefficients $(\{\omega_n\}, \{\alpha_n\})$. The main step of the proof consists of precise estimate of the error terms.

8.6. Some concrete examples

graphs	IFS	vacuum state	deformed vacuum state
Hamming graphs $H(d, N)$	$\omega_n = n$ (Boson)	Gaussian $(N/d \to 0)$ Poisson $(N/d \to \lambda^{-1} > 0)$	Gaussian or Poisson
Johnson graphs $J(v, d)$	$\omega_n = n^2$	exponential $(2d/v \to 1)$ geometric $(2d/v \to p \in (0,1))$	'Poissonization' of exponential distribution
odd graphs O_k	$\omega_{2n-1} = n$ $\omega_{2n} = n$	two-sided Rayleigh	?
homogeneous trees $\mathcal{T}_\kappa$	$\omega_n = 1$ (free)	Wigner semicircle	free Poisson
integer lattices $\mathbb{Z}^N$	$\omega_n = n$ (Boson)	Gaussian	Gaussian
symmetric groups $\mathfrak{S}_n$ (Coxeter)	$\omega_n = n$ (Boson)	Gaussian	Gaussian
Coxeter groups (Fendler)	$\omega_n = 1$ (free)	Wigner semicircle	free Poisson
Spidernets $S(a, b, c)$	$\omega_1 = 1$ $\omega_2 = \cdots = q$	free Meixner law	free Meixner law

For the deformed vacuum states see Section 14.

9. Graph Products

A binary operation of graphs $(G_1, G_2) \mapsto G = \Phi(G_1, G_2)$, which gives rise to a binary operation of adjacency matrices:

$$(A_1, A_2) \mapsto A = \Phi(A_1, A_2), \tag{9.1}$$

would be interesting for the study of growing graphs. Below we focus on some binary operations called *graph products* in general. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with adjacency matrices A_1 and A_2, respectively.

Definition 9.1. The *Cartesian product* $G_1 \times_C G_2$ of two graphs G_1 and G_2 is the graph on $V = V_1 \times V_2$, where two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent whenever (i) $x_1 \sim x_2$ and $y_1 = y_2$; or (ii) $x_1 = x_2$ and $y_1 \sim y_2$.

The adjacency matrix of the Cartesian product $G_1 \times_C G_2$ is given by

$$A[G_1 \times_C G_2] = A_1 \otimes I_2 + I_1 \otimes A_2 \,, \tag{9.2}$$

where I_i denotes the identity matrix with index set $V_i \times V_i$ for $i = 1, 2$.

Definition 9.2. The *Kronecker product* $G_1 \times_K G_2$ of two graphs G_1 and G_2 is the graph on $V = V_1 \times V_2$, where two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent whenever $x_1 \sim x_2$ and $y_1 \sim y_2$.

The adjacency matrix of the Kronecker product $G_1 \times_K G_2$ is given by

$$A[G_1 \times_K G_2] = A_1 \otimes A_2 \,. \tag{9.3}$$

We see that the Kronecker product is a subgraph of the distance-2 graph of the Cartesian product $G_1 \times_C G_2$.

Definition 9.3. The *strong product* $G_1 \times_S G_2$ of two graphs G_1 and G_2 is the graph on $V = V_1 \times V_2$, where two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent whenever (i) $x_1 = x_2$ or $x_1 \sim x_2$; and (ii) $y_1 = y_2$ or $y_1 \sim y_2$.

The adjacency matrix of the strong product $G_1 \times_S G_2$ satisfies

$$A[G_1 \times_S G_2] = A_1 \otimes I_2 + I_1 \otimes A_2 + A_1 \otimes A_2 \,. \tag{9.4}$$

Hence, the edge set of the strong product is the union of those of the Cartesian product and of the Kronecker product.

Definition 9.4. The *lexicographic product* $G_1 \rhd_L G_2$ of two graphs G_1 and G_2 is the graph on $V = V_1 \times V_2$, where two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent whenever (i) $x_1 \sim x_2$; or (ii) $x_1 = x_2$ and $y_1 \sim y_2$.

The adjacency matrix of the lexicographic product $G_1 \rhd_L G_2$ satisfies

$$A[G_1 \rhd_L G_2] = A_1 \otimes J_2 + I_1 \otimes A_2 \,, \tag{9.5}$$

where J_2 is the matrix with index set $V_2 \times V_2$ whose entries are all one.

Definition 9.5. Let $o_2 \in V_2$ be a distinguished vertex of G_2. The *comb product* $G_1 \rhd_{o_2} G_2 = G_1 \rhd G_2$ is the graph on $V = V_1 \times V_2$, where two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent whenever (i) $x_1 \sim x_2$ and $y_1 = y_2 = o_2$; or (ii) $x_1 = x_2$ and $y_1 \sim y_2$.

The adjacency matrix of the comb product $G_1 \rhd_{o_2} G_2$ satisfies

$$A[G_1 \rhd_{o_2} G_2] = A_1 \otimes P_2 + I_1 \otimes A_2 \,, \tag{9.6}$$

where P_2 is the matrix with index set $V_2 \times V_2$ defined by $(P_2)_{xy} = \delta_{xo_2}\delta_{yo_2}$ for $x, y \in V_2$. In other words, P_2 is the rank one projection onto the space spanned by e_{o_2}.

Definition 9.6. For $i = 1, 2$ let $o_i \in V_i$ be a distinguished vertex of G_i. The *star product* $G_1 \star G_2 = (G_1, o_1) \star (G_2, o_2)$ is the graph on $V = V_1 \times V_2$, where two distinct vertices (x_1, y_1) and (x_2, y_2) are adjacent whenever (i) $x_1 \sim x_2$, $y_1 = y_2 = o_2$; or (ii) $x_1 = x_2 = o_1$, $y_1 \sim y_2$.

The adjacency matrix of the star product $G_1 \star G_2$ satisfies

$$A[G_1 \star G_2] = A_1 \otimes P_2 + P_1 \otimes A_2 \,, \tag{9.7}$$

where P_i is the matrix with index set $V_i \times V_i$ defined by $(P_i)_{xy} = \delta_{xo_i}\delta_{yo_i}$ for $x, y \in V_i$. In view of the fact that every vertex (x, y) which does not belong to

$$V_1 \star V_2 = \{(x, o_2)\,;\, x \in V_1\} \cup \{(o_1, y)\,;\, y \in V_2\}$$

is isolated, the induced subgraph of $G_1 \star G_2$ spanned by $V_1 \star V_2$ is also referred to as the star product. Note that the induced subgraph of $G_1 \star G_2$ spanned by $V_1 \star V_2$ coincides with the induced subgraph of $G_1 \times_C G_2$ spanned by $V_1 \star V_2$.

Let $G = G_1 \sharp G_2$ be one of the graph products introduced above. We say that the product is *commutative* if $G_1 \sharp G_2 \cong G_2 \sharp G_1$ and *associative* if $(G_1 \sharp G_2) \sharp G_3 \cong G_1 \sharp (G_2 \sharp G_3)$. The following table summarizes these properties.

graph product	$\sharp$	commutativity	associativity
Cartesian	$\times_C$	yes	yes
Kronecker	$\times_K$	yes	yes
strong	$\times_S$	yes	yes
lexicographic	$\rhd_L$	no	yes
comb	$\rhd$	no	yes
star	$\star$	yes	yes

The terminologies of graph products are not unified in literatures. Our definitions are basically after the books by Balakrishnan–Ranganathan [5] and by Godsil–Royle [18].

10. Cartesian Products and Commutative Independence

To avoid confusion we need precise definitions. Let $\mathcal{A}$ be a unital $*$-algebra. A subset $\mathcal{B} \subset \mathcal{A}$ is called a *$*$-subalgebra* if it is closed under the algebraic operations and the involution, and is called a *unital $*$-subalgebra* if it is a $*$-subalgebra containing $1_{\mathcal{A}}$, the multiplication unit of $\mathcal{A}$.

Definition 10.1. Let $(\mathcal{A}, \varphi)$ be an algebraic probability space. A family $\{\mathcal{A}_\lambda\}$ of unital $*$-subalgebrasof $\mathcal{A}$ is called *commutative independent* or *tensor independent* with respect to φ if

$$\varphi(a_1 \cdots a_m), \qquad a_i \in \mathcal{A}_{\lambda_i}\,, \qquad \lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_m\,,$$

satisfies the following reduction process:

$$\varphi(a_1 \cdots a_m) = \begin{cases} \varphi(a_1)\varphi(a_2 \cdots a_m), & \lambda_1 \notin \{\lambda_2, \ldots, \lambda_m\}, \\ \varphi(a_2 \cdots a_{r-1}(a_1 a_r) a_{r+1} \cdots a_m), & \text{otherwise}, \end{cases}$$

where $r \geq 3$ is the smallest number such that $\lambda_1 = \lambda_r$. Note that neither $\mathcal{A}_\lambda$ nor $\mathcal{A}$ is assumed to be commutative.

Definition 10.2. Let $(\mathcal{A}, \varphi)$ be an algebraic probability space. Let $\{a_\lambda\}$ be a set of random variables and $\mathcal{A}_\lambda$ the unital $*$-subalgebra generated by a_λ and $1_{\mathcal{A}}$. We say that $\{a_\lambda\}$ is *commutative independent* if so is $\{\mathcal{A}_\lambda\}$.

Theorem 10.3 (Commutative CLT). *Let $a_n = a_n^*$ be a sequence of real random variables in an algebraic probability space $(\mathcal{A}, \varphi)$, normalized as $\varphi(a_n) = 0$ and $\varphi(a_n^2) = 1$. Assume that $\{a_n\}$ has uniformly bounded mixed moments, i.e.,*

$$C_m = \sup\{|\varphi(a_{n_1} \cdots a_{n_m})| \,;\, n_1, \ldots, n_m \geq 1\} < \infty, \qquad m \geq 1.$$

If $\{a_n\}$ are commutative independent, we have

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{N} a_n \xrightarrow{\text{m}} g,$$

where g is a random variable obeying the standard normal distribution $N(0,1)$.

The above result is an algebraic version of the classical central limit theorem. The key step of the proof is, after expansion to select the terms $\varphi(a_{n_1} \cdots a_{n_m})$ contributing to the limit. In fact, for an odd m there are no contributing terms, and for an even m we may show that

$$\lim_{N\to\infty} \frac{1}{N^m} \sum \varphi(a_{n_1} \cdots a_{n_{2m}}) = \frac{(2m)!}{2^m m!} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^{2m} e^{-x^2/2} dx, \quad m \geq 0.$$

It is noted that in Theorem 10.3, the moment convergence implies weak convergence because $N(0,1)$ is the solution to a determinate moment problem.

Theorem 10.4. *For $i = 1, 2$ let G_i be a graph, A_i the adjacency matrix, and φ_i a state on the adjacency algebra $\mathcal{A}(G_i)$. Then, being regarded as a random variable in the algebraic probability space $(\mathcal{A}(G_1) \otimes \mathcal{A}(G_2), \varphi_1 \otimes \varphi_2)$, the adjacency matrix $A = A[G_1 \times_C G_2]$ admits an expression*

$$A = A_1 \otimes I_2 + I_1 \otimes A_2 \,,$$

where the right-hand side is a sum of commutative independent random variables. Therefore, letting μ and ν be the spectral distribution of A_1 and A_2, the spectral distribution of A in the state $\varphi_1 \otimes \varphi_2$ is given by the classical convolution $\mu * \nu$.

Theorem 10.5 (CLT for Cartesian powers). *Let $G = (V, E)$ be a graph with adjacency matrix A, and φ a state on $\mathcal{A}(G)$. For $n \geq 1$ let $G_n = G \times_C \cdots \times_C G$ denote the n-fold Cartesian power of G and A_n the adjacency matrix of G_n. Then,*

$$\frac{A_n - n\varphi(A)}{\sqrt{n(\varphi(A^2) - \varphi(A)^2)}} \xrightarrow{\text{m}} g \sim N(0,1), \tag{10.1}$$

in the n-fold product state $\varphi_n = \varphi \otimes \cdots \otimes \varphi$ (n times). Moreover, the normalized spectral distribution of A_n in φ_n converges to $N(0,1)$ weakly.

11. Kronecker and strong products

The Mellin convolution is originally defined on the basis of multiplicative structure of $\mathbb{R}_{>0} = (0, \infty)$. For our purpose, extending slightly the definition, we define the *Mellin convolution* of two probability measures on $\mathbb{R}$ to be a probability measure $\mu *_M \nu$ uniquely specified by

$$\int_{\mathbb{R}} f(z) \mu *_M \nu(dz) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(xy) \mu(dx) \nu(dy), \qquad f \in C_b(\mathbb{R}). \tag{11.1}$$

By definition we have $\delta_a *_M \delta_b = \delta_{ab}$ for $a, b \in \mathbb{R}$. Furthermore, $\delta_0 *_M \mu = \delta_0$ and $\delta_1 *_M \mu = \mu$ for any probability measure μ.

Theorem 11.1. *For $i = 1, 2$ let G_i be a graph and A_i the adjacency matrix. Let φ_i be a state on the adjacency algebra $\mathcal{A}(G_i)$ and μ_i the spectral distribution of A_i in φ_i. Then, the adjacency matrix of the Kronecker product $G_1 \times_K G_2$, regarded as a random variable in $(\mathcal{A}(G_1) \otimes \mathcal{A}(G_2), \varphi_1 \otimes \varphi_2)$, admits an expression*

$$A[G_1 \times_K G_2] = A_1 \otimes A_2 \tag{11.2}$$

and its spectral distribution is given by $\mu_1 *_M \mu_2$.

Theorem 11.2. *Notations and assumptions being the same as in Theorem 11.2, the adjacency matrix of the strong product $G_1 \times_S G_2$, regarded as a random variable in $(\mathcal{A}(G_1) \otimes \mathcal{A}(G_2), \varphi_1 \otimes \varphi_2)$, admits an expression*

$$A[G_1 \times_S G_2] = A_1 \otimes I_2 + I_1 \otimes A_2 + A_1 \otimes A_2 \tag{11.3}$$

and its spectral distribution is given by $S^{-1}(S\mu_1 *_M S\mu_2)$, where S is the shift defined by $S\mu(dx) = \mu(dx - 1)$.

12. Lexicographic and Comb Products, and Monotone Independence

Let $(\Lambda, <)$ be a totally ordered set. In a finite sequence

$$\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_s \neq \cdots \neq \lambda_m\,, \quad \lambda_i \in \Lambda, \quad m \geq 2, \tag{12.1}$$

λ_s is called a *peak* if (i) $1 < s < m$, $\lambda_{s-1} < \lambda_s$ and $\lambda_s > \lambda_{s+1}$; or (ii) $s = 1$ and $\lambda_1 > \lambda_2$; or (iii) $s = m$ and $\lambda_{m-1} < \lambda_m$.

Definition 12.1. Let $(\mathcal{A}, \varphi)$ be an algebraic probability space and let $\{\mathcal{A}_\lambda\,;\, \lambda \in \Lambda\}$ be a set of $*$-subalgebras of $\mathcal{A}$ indexed by a totally ordered set $(\Lambda, <)$. We say that $\{\mathcal{A}_\lambda\}$ is *monotone independent* if

$$\varphi(a_1 \cdots a_s \ldots a_m) = \varphi(a_s)\varphi(a_1 \cdots \check{a}_s \cdots a_m), \qquad \check{a}_s\text{: omission},$$

for any $a_i \in \mathcal{A}_{\lambda_i}$, where $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_m$ and λ_s is a peak.

Definition 12.2. Let $(\mathcal{A}, \varphi)$ be an algebraic probability space. Let $\{a_\lambda\}$ be a set of random variables indexed by a totally ordered set $(\Lambda, <)$, and $\mathcal{A}_\lambda$ the $*$-subalgebra generated by a_λ. We say that $\{a_\lambda\}$ is *monotone independent* if so is $\{\mathcal{A}_\lambda\}$.

The concept of monotone independence is due to Lu [40] and Muraki [42].

Theorem 12.3. *For $i = 1, 2$ let G_i be a graph and A_i the adjacency matrix. Let φ_1 be an arbitrary state on $\mathcal{A}(G_1)$ and φ_2 the vacuum state on $\mathcal{A}(G_2)$ at a distinguished vertex o_2 in G_2. Then, being regarded as a random variable in $(\mathcal{A}(G_1) \otimes \mathcal{A}(G_2), \varphi_1 \otimes \varphi_2)$, the adjacency matrix $A = A[G_1 \rhd_{o_2} G_2]$ admits an expression*

$$A = A_1 \otimes P_2 + I_1 \otimes A_2, \tag{12.2}$$

where the right-hand side is a sum of monotone independent random variables.

Theorem 12.4. *For $i = 1, 2$ let G_i be a graph and A_i the adjacency matrix. Let φ_1 be an arbitrary state on $\mathcal{A}(G_1)$. Assume that G_2 is a finite graph, and let φ_2 be the state on $\mathcal{A}(G_2)$ defined by the density matrix $J_2/|V_2|$, where J_2 is the matrix whose entries are all one. Then, being regarded as a random variable in $(\mathcal{A}(G_1) \otimes \mathcal{A}(G_2), \varphi_1 \otimes \varphi_2)$, the adjacency matrix $A = A[G_1 \rhd_L G_2]$ admits an expression*

$$A = A_1 \otimes J_2 + I_1 \otimes A_2, \tag{12.3}$$

where the right-hand side is a sum of monotone independent random variables.

Theorem 12.5 (Monotone CLT)**.** *Let $a_n = a_n^*$ be a sequence of real random variables in an algebraic probability space $(\mathcal{A}, \varphi)$, normalized as $\varphi(a_n) = 0$ and $\varphi(a_n^2) = 1$. Assume that $\{a_n\}$ has uniformly bounded mixed moments, see Theorem 10.3. If $\{a_n\}$ are monotone independent, we have*

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{N} a_n \xrightarrow{\mathrm{m}} \alpha, \tag{12.4}$$

where α is a random variable obeying the normalized arcsine law.

As in the commutative CLT (Theorem 10.3), the main step of the proof is to select contributing terms $\varphi(a_{n_1} \cdots a_{n_m})$ after the expansion. In fact, for an odd m there are no contributing terms and for an even m, we may show that

$$\lim_{N\to\infty} \frac{1}{N^m} \sum \varphi(a_{n_1} \cdots a_{n_{2m}}) = \frac{(2m)!}{2^m m! m!} = \frac{1}{\pi} \int_{-\sqrt{2}}^{\sqrt{2}} \frac{x^{2m}}{\sqrt{2 - x^2}}\, dx, \quad m \geq 0.$$

For the details see e.g., Hora–Obata [30], Muraki [43], Saigo [54].

For monotone independent real random variables $a = a^*$ and $b = b^*$ in an algebraic probability space $(\mathcal{A}, \varphi)$, we are interested in the spectral distribution of $a + b$. In fact, the spectral distribution of $a + b$ is given by the monotone convolution due to Muraki [42, 43], see also Hasebe [20].

13. Star Products and Boolean Independence

Definition 13.1. Let $(\mathcal{A}, \varphi)$ be an algebraic probability space and $\{\mathcal{A}_\lambda\,;\, \lambda \in \Lambda\}$ be a set of $*$-subalgebras of $\mathcal{A}$. We say that $\{\mathcal{A}_\lambda\}$ is *Boolean independent* if

$$\varphi(a_1 \cdots a_m) = \varphi(a_1)\varphi(a_2 \cdots a_m)$$

for any $a_i \in \mathcal{A}_{\lambda_i}$, where $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_m$.

The concept of Boolean independence was introduced by Speicher–Woroudi [56], while the idea appeared implicitly in Bożejko [10].

Theorem 13.2 (Boolean CLT)**.** *Let $a_n = a_n^*$ be a sequence of real random variables in an algebraic probability space $(\mathcal{A}, \varphi)$, normalized as $\varphi(a_n) = 0$ and $\varphi(a_n^2) = 1$. Assume that $\{a_n\}$ has uniformly bounded mixed moments, see Theorem 10.3. If $\{a_n\}$ are Boolean independent, for any $m \geq 0$ we have*

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{N} a_n \xrightarrow{\mathrm{m}} \beta \tag{13.1}$$

where β is a random variable obeying the normalized Bernoulli distribution.

The proof is similar to the commutative and monotone CLTs, for the details see e.g., Hora–Obata [30], Obata [45], Speicher–Woroudi [56].

Theorem 13.3. *For $i = 1, 2$ let G_i be a graph with a distinguished vertex o_i, and A_i the adjacency matrix. Let φ_i be the vacuum state on $\mathcal{A}(G_i)$ at o_i. Then, being regarded as a random variable in $(\mathcal{A}(G_1) \otimes \mathcal{A}(G_2), \varphi_1 \otimes \varphi_2)$, the adjacency matrix $A = A[(G_1, o_1) \star (G_2, o_2)]$ admits an expression*

$$A = A_1 \otimes P_2 + P_1 \otimes A_2, \tag{13.2}$$

where the right-hand side is a sum of Boolean independent random variables.

For two probability measures μ and ν in $\mathfrak{P}_{\mathrm{fm}}(\mathbb{R})$ we may take Boolean independent algebraic realizations a and b. If the spectral distribution of $a+b$ is uniquely determined (e.g., compactly supported), it is called the *Boolean convolution* of μ and ν, and is denoted by $\mu \uplus \nu$. The Boolean convolution is calculated through the moment generating functions.

14. Relevant topics and further applications

14.1. A generalization of interacting Fock space

Our discussion has been so far based on the actions of B^ϵ defined by Jacobi coefficients, see Figure 1. It is interesting to start with a tridiagonal operator $T = T^+ + T^- + T^\circ$, where T^ϵ is defined as in Figure 4 with $a_n, b_n, c_n \in \mathbb{R}$, $a_n \neq 0$ and $c_n \neq 0$. Note that $(T^-)^* = T^+$ is no longer assumed.

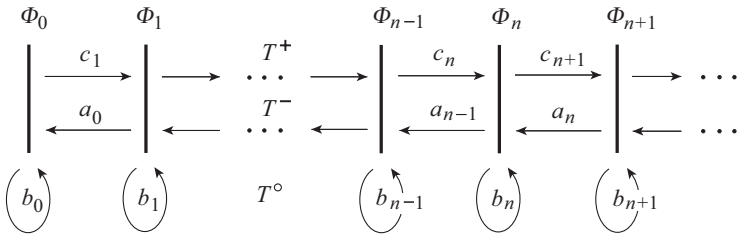

Figure 4: The actions of T^+, T^-, T°

Since the combinatorial argument remain valid, the Accardi–Bożejko formula [2] holds after obvious modification.

Theorem 14.1. *For $m \geq 0$ we have*

$$\langle \Phi_0, T^m \Phi_0 \rangle = \sum_{\vartheta \in \mathcal{P}_{\mathrm{NCPS}}(m)} \prod_{\substack{v \in \vartheta \\ |v|=1}} b(d_\vartheta(v)-1) \prod_{\substack{v \in \vartheta \\ |v|=2}} a(d_\vartheta(v)-1)\, c(d_\vartheta(v)).$$

Let $\{Q_n(x)\}$ be the polynomials defined inductively as

$$\begin{aligned} Q_0(x) &= 1, \\ xQ_0(x) &= a_0 Q_1(x) + b_0 Q_0(x), \\ xQ_n(x) &= a_n Q_{n+1}(x) + b_n Q_n(x) + c_n Q_{n-1}(x), \qquad n \geq 1. \end{aligned}$$

We set

$$P_n(x) = a_0 a_1 \cdots a_{n-1} Q_n(x), \qquad n \geq 0.$$

Then we have

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x - \alpha_1, \\ xP_n(x) &= P_{n+1}(x) + \alpha_{n+1} P_n(x) + \omega_n P_{n-1}(x), \qquad n \geq 1, \end{aligned}$$

where

$$\omega_n = a_{n-1} c_n, \qquad \alpha_n = b_{n-1}\,, \qquad n \geq 1.$$

We are interested in the case where $(\{\omega_n\}, \{\alpha_n\})$ are Jacobi coefficients.

Theorem 14.2. *Notations and assumptions being as above, assume in addition that* $(\{\omega_n\}, \{\alpha_n\})$ *are Jacobi coefficients and let* μ *be the corresponding probability measure on* $\mathbb{R}$*. Then for any* $i \geq 0$ *and* $j \geq 0$ *we have*

$$\langle \Phi_j, T^m \Phi_i \rangle = \frac{1}{\pi_i} \int_{\mathbb{R}} x^m Q_i(x) Q_j(x) \mu(dx), \quad m \geq 0, \tag{14.1}$$

where

$$\pi_0 = 1, \qquad \pi_i = \int_{\mathbb{R}} Q_i(x)^2 \mu(dx) = \frac{c_1 c_2 \cdots c_i}{a_0 a_1 \cdots a_{i-1}}\,.$$

In particular,

$$\langle \Phi_0, T^m \Phi_0 \rangle = \int_{\mathbb{R}} x^m \mu(dx) = M_m(\mu), \qquad m \geq 0.$$

The above result is an extension of the famous Karlin–McGregor formula [33], which originally gives an integral expression of the transition probability of a birth-and-death process. The detailed argument is found in Obata [50].

14.2. Distance-k graphs of Cartesian powers

Let $k \geq 1$ be an integer. The *distance-k graph* of a graph $G = (V, E)$ is a graph $G^{[k]}$ on V, where two distinct vertices $x, y \in V$ are adjacent whenever $\partial_G(x, y) = k$. By definition, the adjacency matrix of $G^{[k]}$ coincides with the kth distance matrix A_k of G. Clearly, the distance-1 graph $G^{[1]}$ coincides with G itself. Note that the distance-k graph of a connected graph is not necessarily connected.

Theorem 14.3. *Let* $G = (V, E)$ *be a finite connected graph with* $|V| \geq 2$*. For* $k \geq 1$ *and* $N \geq 1$ *let* $G^{[N,k]}$ *be the distance-k graph of the N-fold Cartesian power* $G_N = G \times_C \cdots \times_C G$ *(N times). Let* $A^{[N,k]}$ *be the adjacency matrix. Then, as a random variable in the algebraic probability space* $(\mathcal{A}(G^{[N,k]}), \varphi_{\mathrm{tr}})$ *we have*

$$\frac{A^{[N,k]}}{N^{k/2}} \xrightarrow{\mathrm{m}} \left(\frac{2|E|}{|V|}\right)^{k/2} \frac{1}{k!} \tilde{H}_k(g), \tag{14.2}$$

where g *is a random variable obeying the standard normal distribution* $N(0, 1)$*, and* $\tilde{H}_k(x)$ *is the monic Hermite polynomial of degree* k*, i.e., the orthogonal polynomial associated to the standard normal distribution.*

The above result is regarded as a polynomial extension of CLT. For $k = 2$ the limit distribution is the normalized χ_1^2*-distribution*, which is the solution to a determinate moment problem. However, it is plausible that the uniqueness does not hold for $k \geq 3$.

Theorem 14.3 is shown by Hibino–Lee–Obata [26] after some partial results by Kurihara–Hibino [36] and Obata [49]. As a parallel result to Theorem 14.3, the asymptotic spectral distribution of the distance-k graphs of free product graphs is obtained by Arizmendi–Gaxiola [4].

14.3. Counting walks

Let $G = (V, E)$ be a graph with adjacency matrix A. For $x, y \in V$ and $m \geq 0$ let $W_m(x, y) = W_m(x, y; G)$ denote the number of m-step walks connecting x and y. Then we have

$$W_m(x, y) = \langle e_x, A^m e_y \rangle = (A^m)_{xy}\,.$$

Fix a distinguished vertex $o \in V$ and equip the adjacency algebra with the vacuum state at o. Then the adjacency matrix A becomes a real random variable, of which the spectral distribution is denoted by μ. We thus come to

$$M_m(\mu) = \int_{\mathbb{R}} x^m \mu(dx) = \langle A^m \rangle_o = W_m(o, o; G), \qquad m \geq 0. \tag{14.3}$$

We write $\mathbb{Z}^2 = \{(x, y)\,;\, x, y \in \mathbb{Z}\}$ merely for the Cartesian product set. Accordingly, the two-dimensional integer lattice $\mathbb{Z} \times_C \mathbb{Z}$ should be understood as the graph on $\mathbb{Z}^2$ with adjacency relation

$$(x_1, y_1) \sim (x_2, y_2) \quad \Longleftrightarrow \quad \begin{cases} x_2 = x_1 \pm 1, \\ y_2 = y_1, \end{cases} \quad \text{or} \quad \begin{cases} x_2 = x_1, \\ y_2 = y_1 \pm 1. \end{cases}$$

The induced subgraph of $\mathbb{Z} \times_C \mathbb{Z}$ spanned by a subset $D \subset \mathbb{Z}^2$ is denoted by $L[D]$. Such a graph $L[D]$ is called a *restricted lattice*.

We are particularly interested in restricted lattices which admit Kronecker product structure. Recall that the Kronecker product $\mathbb{Z} \times_K \mathbb{Z}$ is the graph on $\mathbb{Z}^2 = \{(u, v)\,;\, u, v \in \mathbb{Z}\}$ with adjacency relation:

$$(u_1, v_1) \sim_K (u_2, v_2) \quad \Longleftrightarrow \quad u_2 = u_1 \pm 1 \quad \text{and} \quad v_2 = v_1 \pm 1. \tag{14.4}$$

Then we have the following

$$\begin{aligned}
&L[\mathbb{Z}^2] \cong (\mathbb{Z} \times_K \mathbb{Z})^o, \\
&L[x \geq y] \cong (\mathbb{Z}_+ \times_K \mathbb{Z})^o, \\
&L[x \geq y \geq -x] \cong (\mathbb{Z}_+ \times_K \mathbb{Z}_+)^o, \\
&L[x \geq y \geq x - (n-1)] \cong (P_n \times_K \mathbb{Z})^o, \quad n \geq 2, \\
&L\begin{bmatrix} 0 \leq x + y \leq k - 1 \\ 0 \leq x - y \leq l - 1 \end{bmatrix} \cong (P_k \times_K P_l)^o, \quad k \geq 2, \quad l \geq 2,
\end{aligned}$$

where the the symbol G^o stands for the connected component containing $o = (0, 0)$.

The Kronecker product structure of a restricted lattice is useful for counting walks. The numbers of walks from the origin 0 to itself in $\mathbb{Z}$ and in $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ are respectively given by

$$W_{2m}(0; \mathbb{Z}) = \binom{2m}{m}, \quad W_{2m}(0; \mathbb{Z}_+) = C_m = \frac{1}{m+1}\binom{2m}{m}, \quad m \geq 0, \tag{14.5}$$

where C_m is the Catalan number, and $W_{2m+1}(0; \mathbb{Z}) = W_{2m+1}(0; \mathbb{Z}_+) = 0$ for all $m \geq 0$. Then we have

$$W_{2m}(o; L[x \geq y]) = C_m \binom{2m}{m} = \frac{1}{m+1}\binom{2m}{m}^2,$$

$$W_{2m}(o; L[x \geq y \geq -x]) = C_m^2 = \frac{1}{(m+1)^2}\binom{2m}{m}^2.$$

Theorem 14.4. *For $D \subset \mathbb{Z}^2$ let $\mu = \mu_D$ be the spectral distribution of the adjacency matrix of the restricted lattice $L[D]$ in the vacuum state at the origin o. In other words, μ satisfies*

$$W_m(o; L[D]) = \int_{\mathbb{R}} x^m \mu(dx), \qquad m \geq 0.$$

The correspondence between domains and spectral distributions is shown in the following table:

D	$\mu = \mu_D$
$\mathbb{Z}^2$	$\alpha * \alpha = \alpha *_M \alpha$
$\{x \geq y\}$	$w *_M \alpha$
$\{x \geq y \geq -x\}$	$w *_M w$
$\{x \geq 0,\, y \geq 0\}$	$w * w$
$\{y \geq 0\}$	$\alpha * w$
$\mathbb{Z}$	α
$\mathbb{Z}_+$	w

where α is the arcsine law with mean 0 and variance 2, and w the normalized Wigner semicircle distribution.

Note also that

$$M_{2m}(\alpha) = \int_{\mathbb{R}} x^{2m}\alpha(x)\,dx = \binom{2m}{m}, \qquad M_{2m+1}(\alpha) = 0, \qquad m \geq 0.$$

and

$$M_{2m}(w) = \int_{\mathbb{R}} x^{2m}w(x)\,dx = C_m = \frac{1}{m+1}\binom{2m}{m}, \quad M_{2m+1}(w) = 0, \quad m \geq 0.$$

The density functions of μ_D are calculated explicitly in terms of elliptic integrals, see Lee–Obata [38] and Obata [52].

14.4. Deformed vacuum states and coherent states

For a connected graph $G = (V, E)$ we denote by $\partial(x, y)$ the graph distance between two vertices $x, y \in V$. Then the matrices

$$D = [\partial(x, y)], \qquad Q = [q^{\partial(x,y)}], \quad -1 \leq q \leq 1,$$

are called the *distance matrix* and Q-matrix, respectively. The *deformed vacuum state* is defined by

$$\langle a \rangle_q = \langle Q\delta_o, a\delta_o \rangle = \sum_{n=0}^{\infty} q^n |V_n|^{1/2} \langle \Phi_n, a\Phi_0 \rangle, \quad a \in \mathcal{A}(G). \tag{14.6}$$

Note that $\langle \cdot \rangle_q$ is a normalized linear function on $\mathcal{A}(G)$ but not necessarily positive.

Now we consider a distance-regular graphs $G = (V, E)$ with intersection array $(\{c_n\}, \{a_n\}, \{b_n\})$. The degree is given by $\kappa = b_0$. It is noteworthy that $QA = AQ$. Moreover, if $Q = [q^{\partial(x,y)}]$ is a positive definite kernel on V, the deformed vacuum state $\langle \cdot \rangle_q$ is positive (i.e., a state in a strict sense) on the adjacency algebra $\mathcal{A}(G)$.

Now consider growing distance-regular graphs $G^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$. Suppose that each $G^{(\nu)}$ is given a deformed vacuum state $\langle \cdot \rangle_q$, where q may depend on ν. In order to get a reasonable limit we need normalization of the adjacency matrices. Since

$$\langle A \rangle_q = q\kappa, \tag{14.7}$$

$$\Sigma_q^2(A) = \langle (A - \langle A \rangle_q)^2 \rangle_q = \kappa(1 - q)(1 + q + qa_1). \tag{14.8}$$

the normalized adjacency matrix is given by

$$\frac{A_\nu - \langle A_\nu \rangle_q}{\Sigma_q(A_\nu)} = \frac{A_\nu^+}{\Sigma_q(A_\nu)} + \frac{A_\nu^-}{\Sigma_q(A_\nu)} + \frac{A_\nu^\circ - q\kappa(\nu)}{\Sigma_q(A_\nu)}\,. \tag{14.9}$$

We assume that under a good scaling balance of ν and q the limits:

$$\omega_n = \lim_{\nu,q} \frac{b_{n-1}(\nu)c_n(\nu)}{\Sigma_q^2(A_\nu)}\,, \qquad \alpha_n = \lim_{\nu,q} \frac{a_{n-1}(\nu) - q\kappa(\nu)}{\Sigma_q(A_\nu)} \tag{14.10}$$

exist and form Jacobi coefficients. Let $\Gamma_{\{\omega_n\}} = (\Gamma, \{\Psi_n\}, B^+, B^-)$ be the associated interacting Fock space. Moreover, assume that the limit

$$c_n = \lim_{\nu,q} q^n |V_n^{(\nu)}|^{1/2} \tag{14.11}$$

exists for all n for which $\{\alpha_n\}$ is defined. Let Υ be the formal sum of vectors defined by

$$\Upsilon = \sum_{n=0}^{\infty} c_n \Psi_n\,.$$

Theorem 14.5 (QCLT for growing distance-regular graphs in the deformed vacuum state)**.** *Notations and assumptions being as above, we have*

$$\lim_{\nu,q} \left\langle \frac{\tilde{A}_\nu^{\epsilon_m}}{\Sigma_q(A_\nu)} \cdots \frac{\tilde{A}_\nu^{\epsilon_1}}{\Sigma_q(A_\nu)} \right\rangle_q = \langle \Upsilon, B^{\epsilon_m} \cdots B^{\epsilon_1} \Psi_0 \rangle \tag{14.12}$$

for any $\epsilon_1, \ldots, \epsilon_m \in \{+, -, \circ\}$ *and* $m \geq 1$.

In the above statement we do not assume that the deformed vacuum state $\langle \cdot \rangle_q$ is positive, but assume that $\Sigma_q^2(A_\nu) > 0$ for normalization. If each deformed vacuum state $\langle \cdot \rangle_q$ on $\mathcal{A}(G^{(\nu)})$ is positive, there exists a probability measure $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbb{R})$ such that

$$\langle \Upsilon, (B^+ + B^- + B^\circ)^m \Psi_0 \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \ldots. \tag{14.13}$$

This μ is the asymptotic spectral distribution of A_ν in the deformed vacuum state that we are interested in. However, derivation of an explicit form of μ from (14.13) seems to be a difficult problem in general.

As a particular case we record the following

Definition 14.6. Let $(\Gamma, \{\Psi_n\}, B^+, B^-)$ be an interacting Fock space associated with a Jacobi sequence $\{\omega_n\}$. The *coherent vector* with parameter $z \in \mathbb{C}$ is a formal sum of vectors defined by

$$\Omega_z = \Psi_0 + \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{\omega_n \cdots \omega_2 \omega_1}}\, \Psi_n\,. \tag{14.14}$$

A coherent state will emerge in various context of limit theorems and play an interesting role in computing asymptotic spectral distributions. Here we only note that Ω_z is a generalized eigenvector of B^- with an eigenvalue z, i.e., $B^-\Omega_z = z\Omega_z$. More precisely,

$$\langle \Omega_z, B^+ \Phi \rangle = \bar{z} \langle \Omega_z, \Phi \rangle, \qquad \Phi \in \Gamma.$$

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