Dadarlat-Pennig による Dixmier-Douady 理論 (twisted K-theory) の一般化について

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概　要

The classical Dixmier-Douady theory describes the structure of continuous trace C*-algebras in terms of the third cohomology of their primitive spectra. In 1989, Rosenberg formulated twisted K-theory in full generality as the K-theory of a continuous trace C*-algebra with its spectrum homeomorphic to a prescribed space and with a prescribed third cohomology class. Since then twisted K-theory has been extensively studied, partly because its relationship with string theory was revealed in the late '90s. On the other hand, in the Elliott program of the classification of nuclear C*-algebras, the importance of a certain class of C*-algebras with very simple structure, now known as strongly self-absorbing C*-algebras, had been recognized among the specialists long before their formal definition was introduced by Toms-Winter in 2007. Recently, a surprising and unexpected application of them was found by Dadarlat-Pennig, who showed that the Dixmier-Douady theory can be generalized to every strongly self-absorbing C*-algebra in that the classical Dixmier-Douady theory is for the trivial C*-algebra, the complex numbers. Moreover, a generalized cohomology theory arises from every strongly self-absorbing C*-algebra, whose characteristic classes have higher terms beyond the third cohomology. In this talk, I will give an account of this theory for non-specialists.

1. Dixmier-Douady theory

1.1. C*-algebras

We first introduce the notion of C*-algebras. The reader is referred to standard textbooks [15] and [21] for the basics of C*-algebras.

Let H be a Hilbert space. We denote by $\mathbb{B}(H)$ the set of bounded operators on H. For $T \in \mathbb{B}(H)$, the operator norm of T is defined by

$$\|T\| = \sup_{\xi \in H\setminus\{0\}} \frac{\|T\xi\|}{\|\xi\|}.$$

Then $\mathbb{B}(H)$ is a Banach space with respect to the operator norm, and at the same time it is an algebra over $\mathbb{C}$. Moreover, it has a $*$-operation assigning the adjoint operator T^* to T, where T^* is characterized by $\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle$ for any $\xi, \eta \in H$.

Definition 1.1. A *C*-algebra* is a subalgebra of $\mathbb{B}(H)$ closed under the norm topology and the $*$-operation.

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In what follows, we assume that H is a separable infinite dimensional Hilbert space. We denote by $\mathbb{K}(H)$, or $\mathbb{K}$ for simplicity, the set of compact operators on H, which is a typical example of a C*-algebra. A C*-algebra A is said to be *stable* if A is isomorphic to $A \otimes \mathbb{K}$. Since $H \otimes H \cong H$, we have $\mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$, and $A \otimes \mathbb{K}$ is always stable for any C*-algebra A.

Although $\mathbb{K}$ often plays important roles in the theory of C*-algebras, $\mathbb{K}$ itself is regarded as a trivial C*-algebra among specialists for various reasons: its structure is well-understood, it has only one irreducible representation, A and $A \otimes \mathbb{K}$ are Morita equivalent, etc. However, being trivial or not of course depends on one's perspective, and the Dixmier-Douady theory shows one of few non-trivial aspects of $\mathbb{K}$.

It is possible to give a Hilbert-space-free definition of C*-algebras: an (abstract) C*-algebra A is a Banach $*$-algebra satisfying the C*-condition $\|T^*T\| = \|T\|^2$ for any $T \in A$. For example, the set of complex valued continuous functions $C(X)$ on a compact Hausdorff space X is a C*-algebra with $*$-operation $f^*(x) = \overline{f(x)}$ and norm $\|f\| = \max_{x\in X} |f(x)|$.

In what follows, we assume that X is a compact metric space, though many statements hold for locally compact and paracompact X. For a C*-algebra A, we denote by $C(X, A)$ the set of continuous A-valued functions on X, which is a C*-algebra with pointwise operations and norm $\|f\| = \max_{x\in X} \|f(x)\|$. This C*-algebra is identified with the tensor product $C(X) \otimes A$ in an appropriate sense.

1.2. Dixmier-Douady theory

The classical Dixmier-Douady theory [10] says that continuous trace C*-algebra A with fixed spectrum X are completely classified by a characteristic class $\delta(A) \in H^3(X, \mathbb{Z})$, called the Dixmier-Douady class, up to stable isomorphism preserving $C(X)$. The reader is referred to [17] for detailed accounts of the Dixmier-Douady theory, and to [20] for more friendly introduction.

A continuous trace C*-algebra A is a continuous field of C*-algebras over a space X with fibers Morita equivalent to $\mathbb{C}$ satisfying the so-called Fell condition. Since the stabilization $A \otimes \mathbb{K}$ of A is always a locally trivial field with fibers isomorphic to $\mathbb{K}$, we avoid the complicated notion and technicality of continuous fields of C*-algebras, the spectrum of a C* algebra, etc, and we focus on locally trivial fields here.

Let $p : E \to X$ be a fiber bundle with fibers $\mathbb{K}$ and structure group $\mathrm{Aut}(\mathbb{K})$. Then the set of continuous sections $s : X \to E$, denoted by $\Gamma(E)$, is a C*-algebra with fiberwise operations and norm $\|s\| = \max_{x\in X} \|s(x)\|$. A stable continuous trace C*-algebra A is nothing but the section algebra $\Gamma(E)$ for some E as above, and we call it a *locally trivial continuous field* of $\mathbb{K}$ over X. The same term is used if $\mathbb{K}$ is replaced with an arbitrary C*-algebra. Note that $\Gamma(E)$ naturally comes with a C*-subalgebra $C(X)$. Whenever we discuss isomorphisms between (locally trivial) continuous fields over X, we assume that they leave $C(X)$ fixed pointwise. We can introduce the Dixmier-Douady class $\delta(A) \in H^3(X, \mathbb{Z})$ for $A = \Gamma(E)$ using the classical theory of fiber bundles. There are two options to do it, via sheaf cohomology theory and via homotopy theory, and we discuss them in order.

We denote by $\mathcal{U}(H)$ the set of unitary operators on H equipped with the strong operator topology. It is known that every automorphism α of $\mathbb{K}$ is implemented by a unitary $U \in \mathcal{U}(H)$, that is, $\alpha(T) = \mathrm{Ad}U(T) = UTU^{-1}$. Therefore we have a short exact sequence of topological groups

$$0 \to \mathbb{T} \to \mathcal{U}(H) \to \mathrm{Aut}(\mathbb{K}) \to 0, \tag{1.1}$$

where $\mathrm{Aut}(\mathbb{K})$ is equipped with the point norm topology. The key fact for the following argument is that the group $\mathcal{U}(H)$ is contractible.

Recall that we have a fiber bundle $p : E \to X$ with $A = \Gamma(E)$. Let $\{V_i\}_{i\in I}$ be an open covering of X, and let $h_i : p^{-1}(V_i) \to V_i \times \mathbb{K}$ be locally trivializations. Then we get the transition functions $\alpha_{ij} : V_i \cap V_j \to \mathrm{Aut}(\mathbb{K})$ satisfying

$$h_i \circ h_j^{-1}(x,T) = (x, \alpha_{ij}(x)(T)), \quad \forall x \in V_i \cap V_j.$$

Passing to a refinement of $\{V_i\}_{i\in I}$ if necessary, we may assume that there exist continuous functions $U_{ij} : V_i \cap V_j \to \mathcal{U}(H)$ satisfying $\alpha_{ij}(x) = \mathrm{Ad}U_{ij}(x)$ for any $x \in V_i \cap V_j$. The cocycle relation $\alpha_{ij}(x) \circ \alpha_{jk}(x) = \alpha_{ik}(x)$ for $x \in V_i \cap V_j \cap V_k$ and the exact sequence (1.1) imply that there exist continuous functions $t_{ijk} : V_i \cap V_j \cap V_k \to \mathbb{T}$ satisfying

$$U_{ij}(x)U_{jk}(x) = t_{ijk}(x)U_{ik}(x).$$

The family of functions $\{t_{ijk}\}$ gives rise to a cohomology class $\delta(A) \in H^2(X, \mathcal{S})$, where $\mathcal{S}$ is the sheaf of the germs of continuous $\mathbb{T}$-valued functions on X, and it is known that $H^2(X, \mathcal{S})$ is isomorphic to the Čech cohomology group $\check{H}^3(X, \mathbb{Z})$. If $\delta(A) = 0$, we may choose $\{V_i\}_{i\in I}$ and U_{ij} so that the cocycle relation $U_{ij}(x)U_{jk}(x) = U_{ik}(x)$ holds for any $x \in V_i \cap V_j \cap V_k$. Since $\mathcal{U}(H)$ is contractive, the cocycle $\{U_{ij}\}$ is a coboundary, which means that the fiber bundle E is trivial, and we get $A \cong C(X, \mathbb{K})$.

Theorem 1.2 ([10])**.** *The isomorphism classes of locally trivial continuous fields of* $\mathbb{K}$ *over* X *is completely classified by the Dixmier-Douady class* $\delta(A)$.

Let B be another locally trivial continuous field of $\mathbb{K}$ over X, that is, B is isomorphic to a section algebra $\Gamma(E')$ for another fiber bundle $p' : E' \to X$ with fibers $\mathbb{K}$ and structure group $\mathrm{Aut}(\mathbb{K})$. Then so is $A \otimes_{C(X)} B$, and we have

$$\delta(A \otimes_{C(X)} B) = \delta(A) + \delta(B).$$

Indeed, the continuous field $A \otimes_{C(X)} B$ is identified with the section algebra $\Gamma(E \otimes E')$, which makes sense as we have $\mathbb{K} \otimes \mathbb{K} \cong \mathbb{K}$, and it is easy to show the additivity of the Dixmier-Douady class with respect to the fiberwise tensor product.

Corollary 1.3. *The set* $\mathfrak{Bun}_X(\mathbb{K})$ *of isomorphism classes of locally trivial fields over* X *with fiber* $\mathbb{K}$ *becomes an abelian group isomorphic to* $H^3(X, \mathbb{Z})$ *under operation of tensor product over* $C(X)$.

Now we discuss the second option, which is more appropriate for our purpose of generalizing the Dixmier-Douady theory. The reader is referred to [8] for the basics of algebraic topology. Recall that $\mathbb{T}$ is a $K(\mathbb{Z},1)$ space in the sense that

$$\pi_i(\mathbb{T}) = \begin{cases} \{0\}, & i \neq 1 \\ \mathbb{Z}, & i = 1 \end{cases},$$

and its universal cover $\mathbb{R}$ is contractible. Since $\mathcal{U}(H)$ is contractible, the exact sequence (1.1) implies

$$\pi_i(\mathrm{Aut}(\mathbb{K})) = \begin{cases} \{0\}, & i \neq 2 \\ \mathbb{Z}, & i = 2 \end{cases}$$

that is, the group $\mathrm{Aut}(\mathbb{K})$ is a $K(\mathbb{Z},2)$ space, and its classifying space $B\mathrm{Aut}(\mathbb{K})$ is a $K(\mathbb{Z},3)$ space. Since the isomorphism classes of principal $\mathrm{Aut}(\mathbb{K})$-bundles over X are completely classified by the homotopy set

$$[X, B\mathrm{Aut}(\mathbb{K})] = [X, K(\mathbb{Z},3)] \cong H^3(X,\mathbb{Z}),$$

we get the Dixmier-Douady classification again.

The main point of the Dadarlat-Pennig theory is that the homotopy set

$$[X, B\mathrm{Aut}(D \otimes \mathbb{K})]$$

gives the first group $E^1_D(X)$ of a generalized cohomology theory $E^*_D(X)$ for every strongly self-absorbing C*-algebra D, and the Dixmier-Douady theory is a special case with $D = \mathbb{C}$.

1.3. Twisted K-theory

The K-theory for C*-algebras is a direct generalization of topological K-theory in the sense that we have $K_*(C(X)) = K^*(X)$. Since $K_*(B \otimes \mathbb{K})$ is canonically isomorphic to $K_*(B)$ for any C*-algebra B, we have $K_*(C(X,\mathbb{K})) = K^*(X)$. A natural question arises: what happens if we replace the trivial continuous field $C(X,\mathbb{K})$ of $\mathbb{K}$ with a locally trivial one?

For $\tau \in H^3(X,\mathbb{Z})$, the twisted K-theory $K^{\tau+*}(X)$ in full generality was first introduced by Rosenberg [18]. He defined it by $K_*(A)$, where A is a locally trivial continuous field of $\mathbb{K}$ over X whose Dixmier-Douady class $\delta(A)$ is τ. As in the case of $K^*(X)$, the twisted K-theory has the Atiyah-Hirzebruch spectral sequence, though differentials are twisted by τ (see [2]).

For those interested in operator algebraic treatment of twisted K-theory and its applications to string theory, we strongly recommend Rosenberg's beautiful monograph [19]. For purely topological treatment, the reader is referred to [11] and references therein.

2. Strongly self-absorbing C*-algebras

2.1. The basics of strongly-self-absorbing C*-algebras

The notion of strongly self-absorbing C*-algebras was introduced by Tom-Winter [22] to single out the class of C*-algebras playing distinguished roles in the classification of nuclear C*-algebras.

Definition 2.1. A unital C*-algebra D is said to be strongly self-absorbing if there exist an isomorphism $\psi : D \to D \otimes D$ and a sequence of unitaries $\{U_n\}_{n=1}^{\infty}$ in $\mathcal{U}(D \otimes D)$ such that for any $T \in D$, we have

$$\lim_{n\to\infty} U_n(T \otimes I)U_n^* = \psi(T).$$

The set of complex numbers $\mathbb{C}$ is an example of a strongly self-absorbing C*-algebra though it is often excluded in the literature. We summarize important properties of strongly self-absorbing C*-algebras from [22], [7], [23], and [6].

Theorem 2.2. *Let D be a strongly self-absorbing C*-algebra.*

(1) *D is simple nuclear either stably finite or purely infinite; if it is stably finite, then it admits a unique trace.*

(2) *$K_0(D)$ has a ring structure with unit $[1]$ given by $[p][q] = [\psi^{-1}(p \otimes q)] \in K_0(A)$ for any projections $p, q \in D$, where ψ is an isomorphism from D onto $D \otimes D$.*

(3) *$K_1(D) = \{0\}$ if D satisfies the UCT.*

(4) $\mathrm{Aut}(D)$ *is contractible.*

(5) $\mathrm{Aut}_0(D \otimes \mathbb{K})$ *has homotopy type of a CW-complex.*

A C*-algebra A is said to be *nuclear* if the algebraic tensor product $A \odot B$ has a unique C*-cross norm for any C*-algebra B. Nuclearity is equivalent to a nice approximation property, and it is closed under various basic operation, such as inductive limit. Every commutative C*-algebra is nuclear. $\mathbb{K}$ and the matrix algebra $\mathbb{M}_n$ are nuclear.

Pure infiniteness and stable finiteness are mutually exclusive properties. A unital C*-algebra is *purely infinite*, if for any $a \in A \setminus \{0\}$ there exist $b, c \in A$ satisfying $bac = 1$. A *trace* τ of a C*-algebra A is a linear functional $\tau : A \to \mathbb{C}$ of norm 1 satisfying $\tau(ab) = \tau(ba)$ and $\tau(a^*a) \geq 0$ for any $a, b \in A$. For a separable unital nuclear simple C*-algebra A, *stable finiteness* is equivalent to having a trace.

Note that ψ in (2) is unique up to approximately unitary equivalence, and the ring structure does not depends on the choice of ψ. The assumption in (3) is harmless because there is no known example of a C*-algebra not satisfying the UCT.

All the known strongly self-absorbing C*-algebras other than $\mathbb{C}$ are in the following list: the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, the UHF algebras of infinite type, the Jiang-Su algebras $\mathcal{Z}$, and the tensor product of the Cuntz algebra $\mathcal{O}_\infty$ and the UHF algebras of infinite type. We briefly describe these examples below.

2.2. The Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$

Let n be an integer greater than 1. The Cuntz algebra $\mathcal{O}_n$ is the universal C*-algebra generated by n isometries $\{S_i\}_{i=1}^n$ obeying the the following two relations

$$S_i^* S_j = \delta_{i,j} 1, \quad \sum_{i=1}^n S_i S_i^* = 1.$$

For $n = \infty$, we define the Cuntz algebra $\mathcal{O}_\infty$ by imposing only the first relation. Their K-groups are

$$K_*(\mathcal{O}_n) \cong \begin{cases} \mathbb{Z}/(n-1)\mathbb{Z}, & * = 0 \\ \{0\}, & * = 1 \end{cases},$$

where $\mathbb{Z}/\infty\mathbb{Z}$ is understood as $\mathbb{Z}$.

The Cuntz algebras are typical examples of *Kirchberg algebras*, separable nuclear simple purely infinite C*-algebras. The Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$ are strongly self-absorbing, and they play special roles in the Kirchberg-Phillips classification theorem of Kirchberg algebras (see [16]). $\mathcal{O}_2$ has trivial K-theory, and it plays the role of 0, that is, we have $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$ for any unital simple separable nuclear C^*-algebra A. On the other hand, $\mathcal{O}_\infty$ is KK-equivalent to $\mathbb{C}$, having the same K-theory as $\mathbb{C}$, and it plays the role of 1 among Kirchberg algebras, that is, we have $\mathcal{O}_\infty \otimes B \cong B$ for any Kirchberg algebra B.

2.3. The UHF algebras of infinite type

Recall that the n by n matrix algebra $\mathbb{M}_n$ is identified with $\mathbb{B}(\mathbb{C}^n)$ and it is a C*-algebra. The UHF algebras are infinite tensor products of matrix algebras. More precisely, let $\{n_k\}_{k=1}^\infty$ be a sequence of integers greater than 1. We set

$$A_m = \mathbb{M}_{n_1} \otimes \mathbb{M}_{n_2} \otimes \cdots \otimes \mathbb{M}_{n_m},$$

which is isomorphic to $\mathbb{M}_n$ with $n = \prod_{k=1}^m n_k$. Embedding A_m into A_{m+1} by $a \mapsto a \otimes 1_{\mathbb{M}_{n_{m+1}}}$, we get an inductive system of C*-algebras $\{A_m\}_{m=1}^\infty$. The norm completion of the inductive limit of the system is a C*-algebra, which is the infinite tensor product of $\{\mathbb{M}_{n_k}\}_{k=1}^\infty$ in the category of C*-algebras. Since $K_1(\mathbb{M}_n) = \{0\}$, the K_1-group is trivial for any UHF algebra. Since $\mathbb{M}_n$ has a trace, so does any UHF algebra, and it is stably finite.

A UHF algebra A is said to be of infinite type if $A \otimes A$ is isomorphic to A. The most important UHF algebra of infinite type is the universal UHF algebra $M_\mathbb{Q}$ obtained by setting $n_k = k!$, which absorbs any UHF algebra by tensor product. The notation $M_\mathbb{Q}$ comes from the fact $K_0(M_\mathbb{Q}) = \mathbb{Q}$. Another example of a UHF algebra of infinite type is M_{n^∞} obtained by setting $n_k = n$ for any k, which has $K_0(M_{n^\infty}) = \mathbb{Z}[\frac{1}{n}]$.

2.4. Jiang-Su algebra $\mathcal{Z}$

The Jiang-Su algebra $\mathcal{Z}$ constructed in [13] is a mysterious stably finite C*-algebra without having projections other than 0 and 1. It is KK-equivalent to $\mathbb{C}$, having $K_0(\mathcal{Z}) = \mathbb{Z}$ and $K_1(\mathcal{Z}) = \{0\}$. It is absorbed by every strongly self-absorbing C*-algebra by tensor product (see [23]). In fact, whether a given C*-algebra absorbs $\mathcal{Z}$ or not is the most important criterion of classifiability of it (see [14]).

3. Dadarlat-Pennig theory

Dadarlat-Pennig [4], [5], [6] developed a Dixmier-Douady theory for an arbitrary strongly self-absorbing C*-algebra D in the manner that the classical Dixmier-Douady theory is a special case with $D = \mathbb{C}$.

Theorem 3.1 ([6]). *Let X be a compact metrizable space, and let D be a strongly self-absorbing C^*-algebra. The set $\mathfrak{Bun}_X(D \otimes \mathbb{K})$ of isomorphism classes of locally trivial fields over X with fiber $D \otimes \mathbb{K}$ becomes an abelian group under operation of tensor product over $C(X)$. Moreover, the group is isomorphic to $E^1_D(X)$, the first group of a generalized connective cohomology theory $E^*_D(X)$ defined by the infinite loop space $B\mathrm{Aut}(D \otimes \mathbb{K})$.*

For a pointed topological space E, we denote by ΩE the loop space of E, that is,

$$\Omega E = \{f \in \mathrm{Map}([0,1], E);\ f(0) = f(1) = *\}.$$

A topological space $E = E_0$ is said to be an *infinite loop space* if there exists a sequence of topological spaces $\{E_n\}_{n=1}^{\infty}$ such that ΩE_n is homotopy equivalent to E_{n-1}. Such a sequence is called an Ω-spectrum, and it gives rise to a generalized cohomology via the homotopy sets $[X, E_n]$. In the above theorem, we have $E_0 = \mathrm{Aut}(D \otimes \mathbb{K})$ and $E_1 = B\mathrm{Aut}(D \otimes \mathbb{K})$.

Note that the E_2-page of the Atiyah-Hirzebruch spectral sequence for $E^*_D(X)$ is given by

$$E_2^{p,q} = H^p(X, E^q_D(pt)) = H^p(X, \pi_{-q}(\mathrm{Aut}(D \otimes \mathbb{K}))),$$

which vanishes for $p < 0$ and for $q > 0$. Thus we need to compute the homotopy groups of $\mathrm{Aut}(D \otimes \mathbb{K})$ to extract information of the generalized cohomology theory $E^*_D(X)$. In fact, Dadarlat [3] computed the homotopy groups of the automorphism groups of the Kirchberg algebras, and Dadarlat-Pennig [6] obtained the following result along the same line. Recall that $K_0(D)$ has a ring structure (see Theorem 2.2). It also has a positive cone $K_0(D)_+$ generated by the classes represented by projections.

Theorem 3.2. *Let D be a strongly self-absorbing C^*-algebra not isomorphic to $\mathbb{C}$. Then*

$$\pi_0(\mathrm{Aut}(D \otimes \mathbb{K})) \cong K_0(D)^{\times}_{+},$$

$$\pi_i(\mathrm{Aut}(D \otimes \mathbb{K})) \cong K_i(D), \quad i \geq 1.$$

If moreover D satisfies the UCT,

$$\pi_{2i}(\mathrm{Aut}(D \otimes \mathbb{K})) \cong \begin{cases} K_0(D)^{\times}_{+}, & i = 0 \\ K_0(D), & i \geq 1 \end{cases},$$

$$\pi_{2i-1}(\mathrm{Aut}(D \otimes \mathbb{K})) \cong \{0\}.$$

We recall the K-groups of known strongly self-absorbing C*-algebras in the table.

	$\mathcal{O}_2$	$\mathcal{O}_\infty$	$\mathcal{O}_\infty \otimes M_{n^\infty}$	$\mathcal{O}_\infty \otimes M_{\mathbb{Q}}$	M_{n^∞}	$M_{\mathbb{Q}}$	$\mathcal{Z}$
$K_0(D)$	$\{0\}$	$\mathbb{Z}$	$\mathbb{Z}[\frac{1}{n}]$	$\mathbb{Q}$	$\mathbb{Z}[\frac{1}{n}]$	$\mathbb{Q}$	$\mathbb{Z}$
$K_0(D)^{\times}_{+}$	$\{0\}$	$\{1, -1\}$	$\mathbb{Z}[\frac{1}{n}]^{\times}$	$\mathbb{Q}^{\times}$	$\mathbb{Z}[\frac{1}{n}]^{\times}_{+}$	$\mathbb{Q}^{\times}_{+}$	$\{1\}$

Since the differentials of the Atiyah-Hirzebruch spectral sequence are known to be torsion operators, we get

Corollary 3.3. (1)

$$\mathfrak{Bun}_X(M_{\mathbb{Q}} \otimes \mathbb{K}) \cong H^1(X, \mathbb{Q}_+^\times) \oplus \bigoplus_{k\geq 1} H^{2k+1}(X, \mathbb{Q}),$$

$$\mathfrak{Bun}_X(\mathcal{O}_\infty \otimes M_{\mathbb{Q}} \otimes \mathbb{K}) \cong H^1(X, \mathbb{Q}^\times) \oplus \bigoplus_{k\geq 1} H^{2k+1}(X, \mathbb{Q}).$$

(2) *If $H^*(X, \mathbb{Z})$ is torsion free,*

$$\mathfrak{Bun}_X(\mathcal{Z} \otimes \mathbb{K}) \cong \bigoplus_{k\geq 1} H^{2k+1}(X, \mathbb{Z}),$$

$$\mathfrak{Bun}_X(\mathcal{O}_\infty \otimes \mathbb{K}) \cong H^1(X, \mathbb{Z}/2\mathbb{Z}) \oplus \bigoplus_{k\geq 1} H^{2k+1}(X, \mathbb{Z}).$$

4. Dynamical realization problem

Let G be a discrete group whose classifying space BG can be chosen to be a finite CW-complex, and let EG be its universal covering space. For a G-action $\alpha : G \to \mathrm{Aut}(A)$ on a C*-algebra A, we can construct a principal $\mathrm{Aut}(A)$-bundle $p : \mathcal{P}_\alpha \to BG$ by $\mathcal{P}_\alpha = (EG \times \mathrm{Aut}(A))/G$, where the G-action on $EG \times \mathrm{Aut}(A)$ is given by

$$g \cdot (x, \gamma) = (gx, \alpha_g \circ \gamma),$$

and p is the projection onto the first coordinate. Note that the principal bundle $\mathcal{P}_\alpha$ has the same information as that of the associated bundle $E = \mathcal{P}_\alpha \times_{\mathrm{Aut}(A)} A$, or equivalently, the section algebra $\Gamma(E)$ together with $C(BG)$. Thus we may abuse the notation to write $[\mathcal{P}_\alpha] \in \mathfrak{Bun}_{BG}(A)$ instead of $[\Gamma(E)]$. We denote by $\mathfrak{ABun}_{BG}(A)$ the collection $\{[\mathcal{P}_\alpha]\}$ of the classes in $\mathfrak{Bun}_{BG}(A)$ coming from G-actions on A.

Assume $A = D \otimes \mathbb{K}$ with a strongly self-absorbing D now. Let β be another G-action on $D \otimes \mathbb{K}$. Then since $(D \otimes \mathbb{K}) \otimes (D \otimes \mathbb{K}) \cong D \otimes \mathbb{K}$, the diagonal action $\alpha \otimes \beta$ is again a G-action on $D \otimes \mathbb{K}$, and in fact we have

$$[\mathcal{P}_{\alpha\otimes\beta}] = [\mathcal{P}_\alpha] + [\mathcal{P}_\beta].$$

Thus $\mathfrak{ABun}_{BG}(D \otimes \mathbb{K})$ forms a subsemigroup of $\mathfrak{Bun}_{BG}(D \otimes \mathbb{K})$.

Question 4.1. *Let G and D be as above. Does $\mathfrak{ABun}_{BG}(D \otimes \mathbb{K})$ coincide with $\mathfrak{Bun}_{BG}(D \otimes \mathbb{K})$?*

It is easy to see that the answer is negative for $D = \mathbb{C}$ in general. For any G-action $\alpha : G \to \mathrm{Aut}(\mathbb{K})$, there exists a projective unitary representation $U : G \to \mathcal{U}(H)$ satisfying $\alpha_g = \mathrm{Ad}U_g$. We get a 2-cocycle $\omega \in Z^2(G, \mathbb{T})$ from $U_g U_h = \omega(g, h)U_{gh}$, which gives an invariant $[\omega] \in H^2(BG, \mathbb{T})$ of the action α. On the other hand, the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{T} \to 0$$

of coefficient modules implies the long exact sequence

$$\cdots \to H^2(BG, \mathbb{T}) \to H^3(BG, \mathbb{Z}) \to H^3(BG, \mathbb{R}) \to \cdots$$

and the Dixmier-Douady class $\delta(\mathcal{P}_\alpha)$ is the image of $[\omega]$ in $H^3(BG,\mathbb{Z})$. Since $\delta(\mathcal{P}_\alpha)$ vanishes in $H^3(BG,\mathbb{R})$, it is always a torsion class.

The situation could be completely different for $D \neq \mathbb{C}$ as the structure of $\mathrm{Aut}(D\otimes\mathbb{K})$ is far richer than that of $\mathrm{Aut}(\mathbb{K})$. Although the following might be too ambitious a conjecture, the author cannot resist mentioning it.

Conjecture 4.2. *Let D be a strongly self-absorbing C^*-algebra satisfying the UCT and $D \neq \mathbb{C}$. Let G be a discrete amenable group with BG a finite CW-complex. Then $\mathfrak{ABun}_{BG}(D\otimes\mathbb{K})$ exhausts $\mathfrak{Bun}_{BG}(D\otimes\mathbb{K})$.*

As a first test case, assume that $D = \mathcal{O}_\infty$ and $\dim BG \leq 3$. In this case, we have a split exact sequence

$$0 \to H^3(G,\mathbb{Z}) \to \mathfrak{Bun}_{BG}(\mathcal{O}_\infty\otimes\mathbb{K}) \to H^1(G,\mathbb{Z}/2\mathbb{Z}) \to 0,$$

and $H^1(G,\mathbb{Z}/2\mathbb{Z})$ is identified with $\mathrm{Hom}(G,\mathrm{Out}(\mathcal{O}_\infty\otimes\mathbb{K}))$. In view of the recent work of Hiroki Matui and the author [12] on poly-$\mathbb{Z}$ group-actions on Kirchberg algebras, the conjecture looks plausible, at least in the case of $\dim BG \leq 3$.

A group G is poly-$\mathbb{Z}$ if there exists a subnormal series

$$\{e\} = G_0 \leq G_1 \leq \cdots \leq G_n = G,$$

satisfying $G_k/G_{k-1} \cong \mathbb{Z}$ for any $k = 1,2,\ldots,n$. The number n is called the Hirsch length of G, often denoted by $h(G)$, which does not depend on the choice of the subnormal series as above, and it coincides with the cohomological dimension of G. In fact, there exists a free cocompact polynomial action of G on $\mathbb{R}^n$ (see [9]), and we can choose $EG = \mathbb{R}^n$ and $BG = \mathbb{R}^n/G$.

Matui and the author classified outer actions of poly-$\mathbb{Z}$ groups G with $h(G) \leq 3$ on any Kirchberg algebras, which verifies the conjecture in the corresponding particular case. For example, every element in

$$\mathfrak{Bun}_{\mathbb{T}^3}(\mathcal{O}_\infty\otimes\mathbb{K}) \cong H^1(\mathbb{T}^3,\mathbb{Z}/2\mathbb{Z}) \oplus H^3(\mathbb{T}^3,\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^3 \oplus \mathbb{Z},$$

comes from a $\mathbb{Z}^3$-action on $\mathcal{O}_\infty\otimes\mathbb{K}$.

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