Delay differential Painlevé equations and difference Nevanlinna theory

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Abstract

Necessary conditions are obtained for certain types of rational delay differential equations to admit a transcendental meromorphic solution of hyper-order less than one. The equations obtained include delay Painlevé equations and equations solved by elliptic functions. Difference analogue of Nevanlinna theory is a central tool in the proofs of the main results. An overview of this theory, as well as some of its applications to difference Painlevé equations, are also presented.

1. Introduction

Existence of large classes of solutions that are meromorphic in the whole complex plane is a rare property for differential equations. According to a classical result due to Malmquist [30], if the first order differential equation

$$f' = R(z, f), \tag{1}$$

where $R(z, f)$ is rational in both arguments, has a transcendental meromorphic solution, then (1) reduces into the Riccati equation

$$f' = a_2 f^2 + a_1 f + a_0 \tag{2}$$

with rational coefficients a_0, a_1 and a_2. In the second-order case Painlevé [35, 36], Fuchs [9] and Gambier [10] classified all equations out of the class

$$f'' = F(z, f, f'),$$

where F is rational in f and f' and analytic in z, which have the Painlevé property. They obtained a list of 50 equations, out of which 44 could either be integrated in terms of known functions, or mapped to another equation within the same list. The remaining six equations are now known as the Painlevé equations. See for instance [5, 8, 17], and references therein, for a comprehensive description of the known properties of these equations.

The existence meromorphic solutions is a more common property in the case of difference equations, than in the case of differential equations. Shimomura [42] proved that the difference equation

$$f(z+1) = P(f(z)),$$

where $P(f(z))$ is a polynomial in $f(z)$ with constant coefficients, always has a transcendental entire solution. On the other hand, Yanagihara [46] showed that the difference equation

$$f(z+1) = R(f(z)),$$

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where $R(f(z))$ is rational in $f(z)$ having constant coefficients, has a non-trivial meromorphic solution, independently of the choice of R. Yanagihara [47] also considered higher order equations and showed, for instance, that the difference equation

$$\alpha_n f(z+n) + \alpha_{n-1} f(z+n-1) + \cdots + \alpha_1 f(z+1) = R(f(z)), \qquad \alpha_1, \ldots, \alpha_n \in \mathbb{C},$$

has a non-trivial meromorphic solution if the degree p of the numerator $P(f(z))$ of the rational function $R(f(z))$ satisfies $p \geq q+2$, where q is the degree of the denominator $Q(f(z))$ and $P(f(z))$ and $Q(f(z))$ have no common factors.

Ablowitz, Halburd and Herbst [1] suggested that the existence of sufficiently many finite-order meromorphic solutions of a difference equation is an analogous to the Painlevé property for difference equations. In order to support this claim they showed, for example, that if the difference equation

$$f(z+1) + f(z-1) = R(z, f(z)), \tag{3}$$

where $R(z, f(z))$ is rational in both arguments, has a transcendental meromorphic solution of finite order, then $\deg_f(R(z, f(z))) \leq 2$. Their results are in line with the earlier results by Yanagihara on the first order equation [46]. He proved that if

$$f(z+1) = R(z, f(z)), \tag{4}$$

where $R(z, f(z))$ is rational in both arguments, has a transcendental meromorphic solution of hyper-order strictly less than one, then $\deg_f(R(z, f(z))) = 1$ and thus (4) reduces exactly into the difference Riccati equation. This is a natural difference analogue of Malmquist's result on differential equations. Halburd and the author [20] showed that if (3), where the right hand side has meromorphic coefficients, has an admissible meromorphic solution f of finite order, then either f satisfies simultaneously a difference Riccati equation, or a linear transformation of (3) reduces it into one in a short list of difference equations which consists of difference Painlevé equations and equations related to them, linear equations and linearizable equations. These results give strong supporting evidence that the approach by Ablowitz, Halburd and Herbst is a good complex analytic difference analogue of the Painlevé property. The finite-order condition was relaxed into hyper-order strictly less than one by Halburd, the author and Tohge [23]. We will give a short overview of these results in the following section.

2. Difference Painlevé equations

We denote by $\mathcal{S}(f)$ the set of all meromorphic functions g such that $T(r,g) = o(T(r,f))$ for all r outside of a set with finite logarithmic measure. We say that a meromorphic solution $f(z)$ of a difference equation is admissible if all coefficients of the equation are in $\mathcal{S}(f)$. In other words, the solution has faster growth than any of the coefficients in the sense of Nevanlinna theory. For instance, all transcendental meromorphic solutions of an equation with rational coefficients are admissible.

Theorem 2.1 ([20, 23]). *If the equation*

$$f(z+1) + f(z-1) = R(z, f(z)), \tag{5}$$

where $R(z, f(z))$ is rational and irreducible in $f(z)$ and meromorphic in z, has an admissible meromorphic solution of hyper-order less than one, then either $f(z)$ satisfies a difference Riccati equation

$$f(z+1) = \frac{p(z+1)f(z) + q(z)}{f(z) + p(z)}, \tag{6}$$

where $p, q \in \mathcal{S}(f)$, or equation (5) can be transformed by a linear change in $f(z)$ to one of the following equations:

$$
\begin{aligned}
f(z+1)+f(z)+f(z-1) &= \frac{\pi_1(z)z+\pi_2(z)}{f(z)}+\kappa_1(z) &\quad (7)\\
f(z+1)-f(z)+f(z-1) &= \frac{\pi_1(z)z+\pi_2(z)}{f(z)}+(-1)^z\kappa_1(z) &\quad (8)\\
f(z+1)+f(z-1) &= \frac{\pi_1(z)z+\kappa_1(z)}{f(z)}+\frac{\pi_2(z)}{f(z)^2} &\quad (9)\\
f(z+1)+f(z-1) &= \frac{\pi_1(z)z+\pi_3(z)}{f(z)}+\pi_2(z) &\quad (10)\\
f(z+1)+f(z-1) &= \frac{(\pi_1(z)z+\kappa_1(z))f(z)+\pi_2(z)}{(-1)^{-z}-f(z)^2} &\quad (11)\\
f(z+1)+f(z-1) &= \frac{(\pi_1(z)z+\kappa_1(z))f(z)+\pi_2(z)}{1-f(z)^2} &\quad (12)\\
f(z+1)f(z)+f(z)f(z-1) &= p(z) &\quad (13)\\
f(z+1)+f(z-1) &= p(z)f(z)+q(z) &\quad (14)
\end{aligned}
$$

where $\pi_k, \kappa_k \in \mathcal{S}(f)$ are arbitrary periodic functions with period k.

Equation (7) arises from the theory of orthogonal polynomials (see e.g. [43]). It also appears in the matrix model approach to the two-dimensional quantum gravity [2, 7]. Equation (7) is widely known as the discrete Painlevé I equation. It has continuous limit $f = -1/2 + \varepsilon^2 u$, $\kappa_1 = -3$, $\pi_1 z + \pi_2 = -(3 + 2\varepsilon^4 t)/4, \varepsilon \to 0$, which may be used to map (7) to the Painlevé I equation $u'' = 6u^2 + t$ [6]. In addition, equation (7) possesses a Lax pair, and it may be integrated by using isomonodromy techniques [7, 37]. Equation (9) is an alternate difference Painlevé I equation [6, 13]. It can also be mapped to the continuous Painlevé I equation by a suitable continuous limit, and its Lax pair has been given in [6]. Equation (10) is a known integrable equation with continuous limits to Painlevé I and IV, and its Lax pair has been given in [16, 12].

Equation (12) was found in connection with unitary matrix models of two-dimensional quantum gravity [38], and it was identified as the difference Painlevé II based on a continuous limit to the continuous Painlevé II equation. Equation (12) was also obtained as a similarity reduction of the discrete mKdV equation [33]. It possesses many special properties, including Lax pairs [27, 37], special Airy-type solutions [28, 41] and discrete Miura and auto-Bäcklund transformations [13].

Many studies on equations (7)–(10) and (12) give strong evidence to suggest that they are all integrable [11]. In addition to possessing many properties indicative of integrability, including Lax pairs, they pass the singularity confinement test and have zero algebraic entropy [15, 25, 34]. They are also a part of the coalescence cascade for the discrete Painlevé equations [13]. Equation (11) is a slight variation of (12). Equation (6) is a difference Riccati equation, and (14) a linear difference equation. Equation (13) is linear in $f(z)f(z-1)$ and possesses finite-order meromorphic solutions of many choices of p. The list of equations (6) – (14) contains all known integrable equations of the form (5) and apparently no non-integrable equations.

3. Delay differential Painlevé equations

Certain reductions of integrable differential-difference equations are known to yield delay differential equations with formal continuous limits to (differential) Painlevé equations. For example, Quispel, Capel and Sahadevan [39] obtained the equation

$$f(z)\,[f(z+1)-f(z-1)]+af'(z)=bf(z), \tag{15}$$

where a and b are constants, as a symmetry reduction of the Kac-van Moerbeke equation. In addition they showed that equation (15) has a formal continuous limit to the first Painlevé equation

$$f''=6f^2+z. \tag{16}$$

Moreover, they obtained an associated linear problem for equation (15) by extending the symmetry reduction to the Lax pair for the Kac-van Moerbeke equation.

Painlevé-type delay differential equations were also considered in Grammaticos, Ramani and Moreira [14] from the point of view of a type of singularity confinement property. More recently, Viallet [45] has introduced a notion of algebraic entropy for such equations.

The order of growth of a meromorphic function f is defined by

$$\rho(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f, and

$$\rho_2(f)=\limsup_{r\to\infty}\frac{\log\log T(r,f)}{\log r}$$

is the hyper-order, also called as the iterated 2-order of f. In the following result [22] we obtain a necessary condition of a class of rational delay differential equations to admit a transcendental meromorphic solution of hyper-order less than one.

Theorem 3.1 ([22]). *Let $f(z)$ be a transcendental meromorphic solution of*

$$f(z+1)-f(z-1)+a(z)\frac{f'(z)}{f(z)}=R(z,f(z))=\frac{P(z,f(z))}{Q(z,f(z))}, \tag{17}$$

where $a(z)$ is a rational function, $P(z,f(z))$ is a polynomial in $f(z)$ having rational coefficients in z, and $Q(z,f(z))$ is a polynomial in $f(z)$ with roots that are non-zero rational functions of z and not roots of $P(z,f(z))$. If the hyper-order of $f(z)$ is less than one, then

$$\deg_f(P)=\deg_f(Q)+1\le 3 \quad \textit{or} \quad \deg_f(R)\le 1. \tag{18}$$

We have used the notation $\deg_f(P)=\deg_f(P(z,f(z)))$ for the degree of P as a polynomial in f and $\deg_f(R)=\max\{\deg_f(P),\deg_f(Q)\}$ for the degree of R as a rational function of f.

If $\deg_f(R(z,f(z)))=0$ then equation (17) reduces into

$$f(z+1)-f(z-1)+a(z)\frac{f'(z)}{f(z)}=b(z), \tag{19}$$

where $a(z)$ and $b(z)$ are rational functions. Note that if $b(z)\equiv p\pi i a(z)$, where $p\in\mathbb{N}$, then $f(z)=C\exp(p\pi i z)$, $C\neq 0$, is a one-parameter family of zero-free entire transcendental finite-order solutions of (19) for any rational function $a(z)$. However, equation (19) with an arbitrary rational function $a(z)$, and with $b(z)\equiv p\pi i a(z)$, is not considered to be of Painlevé type unless $a(z)$ is a constant. Therefore the existence of at least one transcendental meromorphic solution having hyper-order less than one is not enough to single out the delay Painlevé equation (15) from within (19) without further assumptions. In the following theorem we will single out the equation (15) from the class (19) by introducing an additional assumption that the meromorphic solution has sufficiently many simple zeros.

In value distribution theory the notation $S(r,f)$ usually means a quantity with the growth $o(T(r,f))$ as $r \to \infty$ outside of an exceptional set of finite linear measure. In what follows we use a modified definition with a larger exceptional set of finite logarithmic measure. We use the notation $N(r,f)$ to denote the integrated counting function of poles counting multiplicities and $\overline{N}(r,f)$ to denote the integrated counting function of poles ignoring multiplicities.

Theorem 3.2. *Let $f(z)$ be a transcendental meromorphic solution of equation* (19)*, where $a(z) \not\equiv 0$ and $b(z)$ are rational functions. If the hyper-order of $f(z)$ is less than one and for any $\epsilon > 0$*

$$\overline{N}\left(r, \frac{1}{f}\right) \geq \left(\frac{3}{4} + \epsilon\right) T(r,f) + S(r,f), \tag{20}$$

then the coefficients $a(z)$ and $b(z)$ are both constants.

In the next theorem we consider an equation outside of the class (17).

Theorem 3.3 ([22])**.** *Let $f(z)$ be a transcendental meromorphic solution of*

$$f(z+1) - f(z-1) = \frac{a(z)f'(z) + b(z)f(z)}{f(z)^2} + c(z), \tag{21}$$

where $a(z) \not\equiv 0$, $b(z)$ and $c(z)$ are rational functions. If the hyper-order of $f(z)$ is less than one and for any $\epsilon > 0$

$$\overline{N}\left(r, \frac{1}{f}\right) \geq \left(\frac{3}{4} + \epsilon\right) T(r,f) + S(r,f), \tag{22}$$

then (21) *has the form*

$$f(z+1) - f(z-1) = \frac{(\lambda + \mu z)f'(z) + (\nu\lambda + \mu(\nu z - 1))f(z)}{f(z)^2}, \tag{23}$$

where λ, μ and ν are constants.

At least in the special case $\mu = \nu = 0$ and $\lambda \neq 0$ the equation (23) has a multi-parameter family of elliptic function solutions

$$f(z) = \alpha\left[\wp(\Omega z; g_2, g_3) - \wp(\Omega; g_2, g_3)\right],$$

where $\wp$ is the Weierstrass elliptic function, Ω, g_2 and g_3 are arbitrary, provided that $\wp'(\Omega; g_2, g_3) \neq 0$ or ∞, and $\alpha^2 = -\lambda\Omega/\wp'(\Omega; g_2, g_3)$. Moreover, when $\mu = 0$, equation (23) has a formal continuous limit to the first Painlevé equation obtained in the following way: We take the limit $\epsilon \to 0$ for fixed $t = \epsilon z$, where $f(z) = 1 - \epsilon^2 y(t)$, $\lambda = 2 + O(\epsilon)$ and $\lambda\nu = -\frac{1}{3}\epsilon^5 + O(\epsilon^6)$. Then equation (23) becomes $\mathrm{d}^3y/\mathrm{d}t^3 = 12y\,\mathrm{d}y/\mathrm{d}t + 1$, which integrates to $\mathrm{d}^2y/\mathrm{d}t^2 = 6y^2 + t - t_0$, for some constant t_0. Replacing t with $z + t_0$ gives the first Painlevé equation (16). Finally, when $\mu = 0$ and $\lambda\nu \neq 0$, equation

(23) is a symmetry reduction of the known integrable differential-difference modified Korteweg-de Vries equation

$$v_t(x,t) = v(x,t)^2 \left(v(x+1,t) - v(x-1,t) \right),$$

in which $v(x,t) = (-2\lambda\nu t)^{-1/2} f(z)$, where $z = x - (2\nu)^{-1} \log t$.

4. Difference Nevanlinna theory

Nevanlinna theory has been applied recently to study meromorphic solutions of complex difference equations [24, 26, 29, 3], and in particular, as we mentioned in the previous sections, to detect integrability in discrete equations [1, 20, 21, 40, 22]. This has lead to development of difference counterparts of many central results of Nevanlinna theory as more efficient tools to study difference equations. The development of this difference Nevanlinna theory has enabled us to further understand the value distribution of meromorphic functions also without any connections to difference equations.

In what follows we will give a brief outline of some of the key result in the difference Nevanlinna theory without presenting proofs of the theorems.

The first result in this section is a difference analogue of the lemma on the logarithmic derivative. For finite-order meromorphic functions this results was obtained by Halburd and the author [18, 19], and, independently, by Chiang and Feng [3]. The result was extended by Halburd, Tohge and the author [23] for meromorphic functions of hyper-order less than one.

Theorem 4.1 ([23])**.** *Let $f(z)$ be a meromorphic function of hyper-order less than one, let $\varepsilon > 0$, and let $c \in \mathbb{C}$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\rho_2(f)-\varepsilon}}\right)$$

as $r \to \infty$ outside of a set of finite logarithmic measure.

The following lemma turns out to be useful in dealing with shifts in Nevanlinna characteristic and counting functions.

Lemma 4.2 ([23])**.** *Let $T : [0,+\infty) \to [0,+\infty)$ be a non-decreasing continuous function and let $s \in (0,\infty)$. If the hyper-order of T is strictly less than one, i.e.,*

$$\limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} = \varsigma < 1 \tag{24}$$

and $\delta \in (0, 1-\varsigma)$ then

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^\delta}\right) \tag{25}$$

where r runs to infinity outside of a set of finite logarithmic measure.

Theorem 4.1 and Lemma 4.2 combined with methods from the Nevanlinna theory yield some rather powerful results on the value distribution of meromorphic solutions of hyper-order less than one of large classes of non-linear difference equations. We state these results first and then give examples of their use by applying them to study meromorphic solutions of difference Painlevé equations.

Let $n \in \mathbb{N}$ and let $c_j \in \mathbb{C}$, $j = 1, \ldots, n$. A difference polynomial in $f(z)$ is a function which is polynomial in $f(z+c_j)$, $j = 1, \ldots, n$, with meromorphic coefficients $a_\lambda(z)$ such that $T(r, a_\lambda) = S(r,f)$ for all λ. The following theorem [18, 23] is a difference analogue of the Clunie Lemma [4].

Theorem 4.3 ([18, 23]). *Let $f(z)$ be a non-constant meromorphic solution of hyper-order less than one of*

$$f(z)^n P(z, f(z)) = Q(z, f(z)),$$

where $P(z, f(z))$ and $Q(z, f(z))$ are difference polynomials in $f(z)$. If the degree of $Q(z, f(z))$ as a polynomial in $f(z)$ and its shifts is at most n, then

$$m\big(r, P(z, f(z))\big) = S(r, f).$$

Laine and Yang [29] have given a generalization of Theorem 4.3 for a larger class of difference equations.

Example 4.4. *In order to demonstrate how Theorem 4.3 can be used to obtain information about the density of poles of meromorphic solutions of hyper-order < 1 of difference equations, we consider as an example the following difference Painlevé equation*

$$f(z+1) + f(z-1) = \frac{\alpha z + \beta}{f(z)} + \frac{\gamma}{f(z)^2} \tag{26}$$

with constant parameters α, β, γ. Suppose that $f(z)$ is a transcendental meromorphic function of hyper-order < 1. Then by considering (26) in the form

$$f(z)^2(f(z+1) + f(z-1)) = (\alpha z + \beta) f(z) + \gamma$$

we may apply Theorem 4.3 with $P(z, f(z)) = f(z+1) + f(z-1)$ and $Q(z, f(z)) = (\alpha z + \beta) f(z) + \gamma$ thus obtaining

$$m(r, f(z+1) + f(z-1)) = S(r, f). \tag{27}$$

Since $T(r, f(z+1) + f(z-1)) = 2T(r, f(z)) + O(\log r)$ by an identity due to Valiron [44] and Mohon'ko [31] and (26), equation (27) yields

$$N(r, f(z+1) + f(z-1)) = 2T(r, f(z)) + S(r, f). \tag{28}$$

Finally, since $N(r, f(z+1) + f(z-1)) \leq 2N(r+1, f(z)) = 2N(r, f(z)) + S(r, f)$ by Lemma 4.2, we conclude by equation (28) that

$$N(r, f(z)) = T(r, f(z)) + S(r, f). \tag{29}$$

In particular, this implies that all transcendental meromorphic solutions of hyper-order < 1 of the difference Painlevé equation (26) have infinitely many poles.

The following theorem enables the analysis of the value distribution of solutions for finite values. It is an analogue of a result due to A. Z. Mohon'ko and V. D. Mohon'ko [32] on differential equations.

Theorem 4.5 ([18, 23]). *Let $f(z)$ be a non-constant meromorphic solution of hyper-order less than one of*

$$P(z, f(z)) = 0 \tag{30}$$

where $P(z, f(z))$ is difference polynomial in $f(z)$. If $P(z, a(z)) \not\equiv 0$ for a meromorphic function $a \in \mathcal{S}(f)$, then

$$m\left(r, \frac{1}{f(z) - a(z)}\right) = S(r, f).$$

Consider again the difference Painlevé equation (26) as an example.

Example 4.6. *If the parameter α is non-zero, then (26) does not have any constant solutions. Therefore Theorem 4.5 yields*

$$m\left(r, \frac{1}{f(z)-a}\right) = S(r,f)$$

for all $a \in \mathbb{C}$, provided that $f(z)$ is of hyper-order < 1 as in Example 4.4. This in particular implies that all transcendental meromorphic solutions of hyper-order < 1 of the non-autonomous equation (26) have infinitely many a-points for all $a \in \mathbb{C}$.

The following theorem is a difference analogue of the second main theorem, where, instead of the usual ramification term involving the derivative of the considered meromorphic function $f(z)$, there is a quantity expressed in terms of the difference operator of $f(z)$. Since periodic functions are the analogues of constants for exact differences, it is natural to consider slowly moving periodic functions as target functions of $f(z)$.

Theorem 4.7 ([18, 23])**.** *Let $c \in \mathbb{C}$, and let f be a meromorphic function of hyper-order less than one such that $\Delta_c f \not\equiv 0$. Let $q \geq 2$, and let $a_1(z), \ldots, a_q(z)$ be distinct meromorphic periodic functions with period c such that $a_k \in \mathcal{S}(f)$ for all $k = 1, \ldots, q$. Then*

$$m(r,f) + \sum_{k=1}^{q} m\left(r, \frac{1}{f-a_k}\right) \leq 2T(r,f) - N_\Delta(r,f) + S(r,f)$$

where

$$N_\Delta(r,f) := 2N(r,f) - N(r,\Delta_c f) + N\left(r, \frac{1}{\Delta_c f}\right).$$

Applications on Theorem 4.7 include difference versions of the deficiency relation and of Picard's theorem. Theorem 4.7 was generalized to holomorphic curves by Halburd, Tohge and the author in [23].

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