# Distortion theorems for holomorphic mappings on bounded symmetric domains

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**Abstract**

In this talk, we will generalize distortion theorems for normalized holomorphic functions on the unit disc in $\mathbb{C}$ to normalized holomorphic mappings on bounded symmetric domains in a higher dimensional complex Banach space.

## 1. Introduction

Let $\mathbb{U} = \{x \in \mathbb{C}; |x| < 1\}$ be the unit disc in $\mathbb{C}$. A holomorphic function $f : \mathbb{U} \longrightarrow \mathbb{C}$ is said to be *normalized* if $f$ satisfies the conditions $f(0) = f'(0) - 1 = 0$. We first recall the (Köbe) distortion theorem for a normalized univalent function on $\mathbb{U}$.

**Theorem 1.1**
*If $f : \mathbb{U} \longrightarrow \mathbb{C}$ is a normalized univalent function, then*

$$\frac{1-|x|}{(1+|x|)^3} \leq |f'(x)| \leq \frac{1+|x|}{(1-|x|)^3}, \quad x \in \mathbb{U}.$$

*The above estimates are sharp.*

It is natural to consider higher dimensional version of the above distortion estimates. We will generalize the condition "normalized" to a complex Banach space.

Let $\mathbb{B}_X = \{z \in X; \|z\| < 1\}$ be the open unit ball in a complex Banach space $X$ with the norm $\|\cdot\|$. A holomorphic mapping $f : \mathbb{B}_X \longrightarrow X$ is said to be *normalized* if $f$ satisfies $f(0) = 0, Df(0) = Id$.

Can we directly extend the above distortion theorem to higher dimensions ? In fact, we can easily find a counter-example in the following two dimensional case.

**Example 1.2**
*Let $f : \mathbb{U}\times\mathbb{U} \longrightarrow \mathbb{C}^2$ be defined by $f(z_1, z_2) := (z_1+200z_2^3, z_2)$. Then $f$ is biholomorphic, and satisfies $f(0,0) = (0,0)$. It follows from $Df(z) = \begin{pmatrix} 1 & 600z_2^2 \\ 0 & 1 \end{pmatrix}$ that $Df(0,0) = Id$. Since $\|z\| = \|(z_1, z_2)\| := \sup\{|z_1|, |z_2|\}$, we have $\|z\| = \frac{1}{2}$ when $z = (0, \frac{1}{2})$. Then,*

$$\frac{1+\|z\|}{(1-\|z\|)^3} = 12 < 150 = |600z_2^2| \leq \sup_{\|y\|=1} \|Df(z)(y)\| = \|Df(z)\|.$$

The above counter-example shows that we can not extend Theorem 1.1 to normalized biholomorphic mappings in higher dimensional complex Banach spaces. So, we are obliged to restrict to proper subclasses of the family of all normalized biholomorphic mappings.

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In this talk, we will generalize distortion theorems for normalized holomorphic functions on the unit disc in $\mathbb{C}$ to normalized holomorphic mappings on bounded symmetric domains in a higher dimensional complex Banach space.

## 2. Preliminaries

Let $X, Y$ be complex Banach spaces. We denote by $L(X, Y)$ the space of continuous linear operators from $X$ into $Y$ with the standard operator norm. Let $I$ be the identity operator in $L(X)$, where $L(X) = L(X, X)$.

Let $\|\cdot\|_X$ be a norm on $X$ and $\|\cdot\|_e$ denote the Euclidean norm on $\mathbb{C}^n$. For $A \in L(X, \mathbb{C}^n)$, let

$$\|A\|_{X,e} = \sup\{\|Az\|_e : \|z\|_X = 1\}$$

and if $X = \mathbb{C}^n$, let

$$\|A\|_X = \sup\{\|Az\|_X : \|z\|_X = 1\}$$

and

$$\|A\|_e = \sup\{\|Az\|_e : \|z\|_e = 1\}.$$

For each $z \in X \setminus \{0\}$, let

$$T(z) = \{l_z \in L(X, \mathbb{C}) : \; l_z(z) = \|z\|_X, \; \|l_z\|_{X,e} = 1\}.$$

This set is nonempty by the Hahn-Banach theorem.

The set of holomorphic mappings from a domain $\Omega \subset X$ into $Y$ is denoted by $H(\Omega, Y)$. The set $H(\Omega, X)$ is denoted by $H(\Omega)$. A mapping $f \in H(\Omega, Y)$ is said to be *biholomorphic* if $f(\Omega)$ is a domain, the inverse $f^{-1}$ exists and is holomorphic on $f(\Omega)$. When $\Omega$ contains the origin, we say that a mapping $f \in H(\Omega)$ is normalized if $f(0) = 0$ and $Df(0) = I$.

The family of normalized biholomorphic mappings in $H(\Omega)$ will be denoted by $S(\Omega)$. In the case of one complex variable, $S(\mathbb{U})$ is the usual family $S$ of normalized univalent functions on the unit disc $\mathbb{U}$. Let $\mathcal{L}S(\Omega)$ be the family of normalized locally biholomorphic mappings of $\Omega$ into $X$. Let $\mathrm{Aut}(\Omega)$ denote the set of biholomorphic automorphisms of $\Omega$. A domain $\Omega$ is said to be *homogeneous* if for any $x, y \in \Omega$, there exists some mapping $f \in \mathrm{Aut}(\Omega)$ such that $f(x) = y$. A point $a \in \Omega$ is called a *symmetric point* if there exists $\sigma_a \in \mathrm{Aut}(\Omega)$ such that $\sigma_a^2 = I_\Omega$ and $a$ is an isolated fix point of $\sigma_a$. A domain $\Omega$ is said to be *symmetric* if all $x \in \Omega$ are symmetric points.

Every bounded symmetric domain in a complex Banach space is homogeneous. Conversely, the open unit ball $\mathbb{B}$ of a Banach space admits a symmetry $\sigma_0(x) := -x$ at 0 and if $\mathbb{B}$ is homogeneous, then $\mathbb{B}$ is a symmetric domain. Banach spaces with a homogeneous open unit ball are precisely the JB*-triples ( see Kaup [41] ).

**Definition 2.1** *A complex Banach space $X$ is called a JB*-triple if there exists a triple product*

$$(x, y, z) \in X \times X \times X \mapsto \{x, y, z\} \in X$$

*satisfying*

*(i) $\{x, y, z\}$ is symmetric bilinear in the outer variables, but conjugate linear in the middle variable,*

*(ii) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$,*

*(iii)* $x \square x \in L(X,X)$ *is a hermitian operator with spectrum* $\geqq 0$,

*(iv)* $\|\{x,x,x\}\| = \|x\|^3$

*for* $a,b,x,y,z \in X$, *where the box operator* $x\square y : X \to X$ *is defined by* $x\square y(\cdot) = \{x,y,\cdot\}$.

**Example 2.2** *(i) A complex Hilbert space* $H$ *with inner product* $\langle\cdot,\cdot\rangle$ *is a JB*-triple with the triple product*

$$\{x,y,z\} = \frac{1}{2}(\langle x,y\rangle z + \langle z,y\rangle x).$$

*(ii) The complex space* $\mathbb{C}^n$ *is also a JB*-triple when it is equipped with the* $\ell_\infty$ *norm* $\|\cdot\|_\infty$ *and the triple product*

$$\{x,y,z\} = (x_i\overline{y_i}z_i)_{1\le i\le n},\ x=(x_i)_{1\le i\le n},\ y=(y_i)_{1\le i\le n},\ z=(z_i)_{1\le i\le n} \in \mathbb{C}^n.$$

*The unit polydisc* $\mathbb{U}^n$ *is the unit ball of* $(\mathbb{C}^n, \|\cdot\|_\infty)$.

We introduce the Bergmann operator $B(a,b) : X \to X$ on a JB*-triple $X$ for $a,b \in X$ defined by

$$B(a,b)(x) = x - 2\{a,b,x\} + \{a,\{b,x,b\},a\} \qquad (x \in X).$$

In particular, the fractional power $B(x,y)^r \in GL(X)$ exists for every $r \in \mathbb{R}$ in a natural way (cf. Kaup [41, p.517]).

Let $\mathbb{B}$ be the unit ball of a JB*-triple $X$. Then, for each $a \in \mathbb{B}$, the Möbius transformation $g_a$ defined by

$$g_a(x) = a + B(a,a)^{1/2}(I_X + x\square a)^{-1}x,$$

is a biholomorphic mapping of $\mathbb{B}$ onto itself with $g_a(0)=a$, $g_a(-a)=0$ and $g_{-a}=g_a^{-1}$. We have

$$Dg_a(0) = B(a,a)^{1/2}, \qquad Dg_{-a}(a) = B(a,a)^{-1/2}.$$

In any dimension, we have

$$\|B(a,a)^{-1/2}\| = \frac{1}{1-\|a\|^2} \tag{2.1}$$

from Kaup [42, Corollary 3.6]. We remark that each $g \in \mathrm{Aut}(\mathbb{B})$ is a composite of a Möbius transformation and a linear isometry, by Cartan's uniqueness theorem, hence it is true that

$$\|[Dg(0)]^{-1}\| = \frac{1}{1-\|a\|^2} \tag{2.2}$$

whenever $g \in \mathrm{Aut}(\mathbb{B})$ satisfies $g(0)=a$.

**Proposition 2.3** *Let* $g_a$ *be as above. Then for any* $a \in \mathbb{B}$, $g_a$ *extends biholomorphically to a neighborhood of* $\overline{\mathbb{B}}$ *and we have*

$$[Dg_a(0)]^{-1}D^2g_a(0)(x,y) = -2\{x,a,y\}, \tag{2.3}$$

$$\|Dg_a(0)\| \le 1, \tag{2.4}$$

$$Dg_{\zeta a}(0) = Dg_a(0), \quad |\zeta|=1, \tag{2.5}$$

$$g_a(a) = \frac{2}{1+\|a\|^2}a, \tag{2.6}$$

$$g_a(x) = x + a - \{x,a,x\} + O(\|a\|^2), \tag{2.7}$$

$$[Dg_a(0)]^{-1} = I_X + O(\|a\|^2). \tag{2.8}$$

*Moreover, we have*

$$\frac{1}{1-\|g_{-z}(w)\|^2} \leq \frac{(1+\|w\|\cdot\|z\|)^2}{(1-\|w\|^2)(1-\|z\|^2)}, \quad z, w \in \mathbb{B}. \tag{2.9}$$

**Lemma 2.4** *Let $\mathbb{B}$ be the open unit ball of a complex Hilbert space $H$ and let $a \in \mathbb{B}$. Then we have*

$$\|B(a,a)^{1/2}\|^2 = \|B(a,a)\| = \begin{cases} (1-\|a\|^2)^2 & \text{if } \dim H = 1 \\ 1-\|a\|^2 & \text{if } \dim H \geq 2. \end{cases}$$

## 3. Distortion results for convex mappings

Let $X$ and $Y$ be Banach spaces and let $G \subset X$ be a convex domain. A biholomorphic mapping $f \in H(G,Y)$ is said to be *convex* if the image $f(G)$ is convex in $Y$. Also, let $K(G)$ be the subfamily of $S(G)$ consisting of convex mappings. We recall the distortion theorem for a normalized convex univalent function on $\mathbb{U}$.

**Theorem 3.1**
*If $f : \mathbb{U} \longrightarrow \mathbb{C}$ is a normalized convex univalent function, then*

$$\frac{1}{(1+|x|)^2} \leq |f'(x)| \leq \frac{1}{(1-|x|)^2}, \quad x \in \mathbb{U}.$$

*The above estimates are sharp.*

About generalization of the above distortion theorem for convex functions, many mathematicians have studied. Gong and Liu [19], Pfaltzgraff and Suffridge [56] extended to the Euclidean balls in $\mathbb{C}^n$. It has been further generalized to the open unit balls of complex Hilbert spaces by Hamada and Kohr [30, 31].

Zhu and Liu [67] have obtained the following distortion theorem for convex mappings $f$ on the open unit balls of complex Banach spaces:

$$\frac{1}{(1+\|x\|)^2} \leq \|Df(x)\| \leq \frac{1+\|x\|}{(1-\|x\|)^2}.$$

Let $G$ be a domain in a Banach space $X$. For each $(x,\xi) \in G \times X$, the infinitesimal Carathéodory pseudometric $\gamma_G(x,\xi)$ on $G$ is defined by,

$$\gamma_G(x,\xi) = \sup\{|Dh(x)\xi| : h \in H(G,\mathbb{U}), h(x) = 0\}.$$

Each $\varphi \in \mathrm{Aut}(G)$ is an isometry in this pseudometric:

$$\gamma_G(x,\xi) = \gamma_G(\varphi(x), D\varphi(x)\xi)$$

and for the open unit ball $\mathbb{B}$ in a Banach space $X$, one has $\gamma_{\mathbb{B}}(0,\xi) = \|\xi\|$. We note that $\gamma_{\mathbb{U}}$ is the Poincaré metric on $\mathbb{U}$ (cf. Dineen [14, p.54]). We will make use of the following distortion theorem in Hamada and Kohr [31, Remark 4] (see also Zhu and Liu [67, Theorem 2.1]).

**Lemma 3.2** *Let $\mathbb{B}$ be the open unit ball in a Banach space $X$ and let $f : \mathbb{B} \to X$ be a normalized convex mapping on $\mathbb{B}$. Then we have*

$$\frac{1-\|x\|}{1+\|x\|}\gamma_{\mathbb{B}}(x,y) \le \|Df(x)y\| \le \frac{1+\|x\|}{1-\|x\|}\gamma_{\mathbb{B}}(x,y) \tag{3.1}$$

*for each $x \in \mathbb{B}$ and $y \in X$.*

**Theorem 3.3** *Let $\mathbb{B}$ be the open unit ball of a JB*-triple $X$. Given a normalized convex mapping $f : \mathbb{B} \to X$ with derivative $Df$, we have, for $a, b, x \in \mathbb{B}$ and $y \in X$,*

(i) $\dfrac{1}{(1+\|x\|)^2} \le \|Df(x)\| \le \dfrac{1}{(1-\|x\|)^2}$;

(ii) $\dfrac{(1-\|x\|)\|y\|}{(1+\|x\|)\|B(x,x)^{1/2}\|} \le \|Df(x)y\| \le \dfrac{\|y\|}{(1-\|x\|)^2}$;

*where $B(x,x) : X \to X$ is the Bergmann operator and $\|B(x,x)^{1/2}\| \le 1$.*

**Remark 3.4** *It has been proved by Zhu and Liu [67] the following distortion theorem for convex mappings on the open unit balls of complex Hilbert spaces:*

$$\frac{\|y\|\sqrt{1-\|x\|^2}}{(1+\|x\|)^2} \le \|Df(x)y\| \le \frac{\|y\|}{(1-\|x\|)^2}. \tag{3.2}$$

*This result is generalized in Theorem 3.3 (ii) to homogeneous balls. In fact, by Lemma 2.4, we have that $\|B(x,x)^{1/2}\| = \sqrt{1-\|x\|^2}$ for complex Hilbert spaces of dimension greater than 1.*

*Zhu and Liu [67, Conjecture 2.2] conjectured that Theorem 3.3 (i) holds for convex mappings on the open unit balls of complex Banach spaces. Our theorem is an affirmative answer to this conjecture for convex mappings on homogeneous balls. Hamada and Kohr [30] proved that the upper bound in Theorem 3.3 (i) is sharp for the open unit balls of complex Hilbert spaces although the lower bound is not sharp for the Euclidean balls of dimension at least 2 (cf. Liczberski [48]).*

## 4. Distortion results for linearly invariant families

We begin this section with the notion of linearly invariant families on the unit ball $\mathbb{B}$ of a complex Banach space $X$. Then we give the notion of norm-order and obtain distortion results for linearly invariant families on the unit ball of finite dimensional JB*-triples (cf. Hamada, Honda and Kohr [26], Pfaltzgraff and Suffridge [56]).

**Definition 4.1** *Let $\mathbb{B}$ be the unit ball of a complex Banach space $X$. Then a family $\mathcal{F}$ is called a linearly invariant family (L.I.F.) if the following conditions hold:*

*(i) $\mathcal{F} \subset \mathcal{L}S(\mathbb{B})$;*

*and*

*(ii) $\Lambda_\phi(f) \in \mathcal{F}$, for all $f \in \mathcal{F}$ and $\phi \in \mathrm{Aut}(\mathbb{B})$.*

*Here $\Lambda_\phi(f)$ is the Koebe transform of $f$ given by*

$$\Lambda_\phi(f)(z) = [D\phi(0)]^{-1}[Df(\phi(0))]^{-1}(f(\phi(z)) - f(\phi(0))), \quad z \in \mathbb{B}.$$

Note that the Koebe transform has the group property $\Lambda_\psi \circ \Lambda_\phi = \Lambda_{\phi\circ\psi}$.

If $X = \mathbb{C}^n$ and $\mathcal{F}$ is a linearly invariant family, we define two types of *norm-order* of $\mathcal{F}$ (cf. Hamada, Honda and Kohr [26], Pfaltzgraff and Suffridge [56]), given by

$$\|\operatorname{ord}\|_{e,1}\mathcal{F} = \sup_{f\in\mathcal{F}} \sup_{\|w\|_X=1} \left\{\frac{1}{2}\|D^2 f(0)(w,\cdot)\|_{X,e}\right\}$$

and

$$\|\operatorname{ord}\|_{e,2}\mathcal{F} = \sup_{f\in\mathcal{F}} \sup_{\|w\|_X=1} \left\{\frac{1}{2}\|D^2 f(0)(w,w)\|_{e}\right\}.$$

It is clear that $\|\operatorname{ord}\|_{e,1}\mathcal{F} \geq \|\operatorname{ord}\|_{e,2}\mathcal{F}$. On the other hand, since

$$D^2 f(0)(z,w) = \frac{1}{2}\Big\{D^2 f(0)(z+w,z+w) - D^2 f(0)(z,z) - D^2 f(0)(w,w)\Big\},$$

we obtain $\|\operatorname{ord}\|_{e,1}\mathcal{F} \leq 3\|\operatorname{ord}\|_{e,2}\mathcal{F}$. Moreover, if $X$ is a finite dimensional complex Hilbert space, then $\|\operatorname{ord}\|_{e,1}\mathcal{F} = \|\operatorname{ord}\|_{e,2}\mathcal{F}$ by Hörmander [38, Theorem 4].

We also define the trace-order of $\mathcal{F}$ (cf. Hamada, Honda and Kohr [27], Pfaltzgraff [52]) given by

$$\operatorname{ord}\mathcal{F} = \sup_{f\in\mathcal{F}} \sup_{\|w\|_X=1} \left\{\frac{1}{2}\left|\operatorname{tr}\left[D^2 f(0)(w,\cdot)\right]\right|\right\}.$$

We now give some examples of linearly invariant families on the unit ball $\mathbb{B}$ of a complex Banach space $X$ (cf. Hamada, Honda and Kohr [26, 27], Pfaltzgraff and Suffridge [55]).

**Example 4.2** *(i) $K(\mathbb{B})$, the set of convex mappings in $\mathcal{LS}(\mathbb{B})$. If $X$ is a finite dimensional complex Hilbert space, then $\|\operatorname{ord}\|_{e,1}K(\mathbb{B}) = 1$ (see Pfaltzgraff and Suffridge [56] and Hamada and Kohr [29]). On the other hand, it is known that in the case of an $n$-dimensional complex Hilbert space with $n \geq 2$, $\operatorname{ord}K(\mathbb{B}) > (n+1)/2$ and $\operatorname{ord}K(\mathbb{B})$ is unknown (see Pfaltzgraff and Suffridge [55]).*

*(ii) $S(\mathbb{B})$, the set of all biholomorphic mappings in $\mathcal{LS}(\mathbb{B})$. If $X$ is a complex Hilbert space of dimension $n$, where $n > 1$, the linearly invariant family $S(\mathbb{B})$ does not have finite trace-order (see Barnard, FitzGerald and Gong [2]; cf. Pfaltzgraff and Suffridge [52]).*

*(iii) $\mathcal{U}_\alpha(\mathbb{B})$, the union of all linearly invariant families contained in $\mathcal{LS}(\mathbb{B})$ with trace-order not greater than $\alpha$. This is a generalization of the universal linearly invariant families $\mathcal{U}_\alpha = \mathcal{U}_\alpha(\Delta)$ considered in Pommerenke [57].*

*(iv) If $\mathcal{G}$ is a nonempty subset of $\mathcal{LS}(\mathbb{B})$, then the linearly invariant family generated by $\mathcal{G}$ is the family*

$$\Lambda[\mathcal{G}] = \{\Lambda_\phi(g) : \ g \in \mathcal{G},\ \phi \in \operatorname{Aut}(\mathbb{B})\}.$$

*The linear invariance is a consequence of the group property of the Koebe transform. Obviously, $\Lambda[\mathcal{G}] = \mathcal{G}$ if and only if $\mathcal{G}$ is a linearly invariant family. In the cases of the unit Euclidean ball and the unit polydisc of $\mathbb{C}^n$, this example provided a useful technique for generating many interesting mappings (see Pfaltzgraff [52], Pfaltzgraff and Suffridge [54, 55]). For example, we can use a single mapping $f$ from $\mathcal{LS}(\mathbb{B})$ to generate the linearly invariant family $\Lambda[\{f\}]$. The family $\Lambda[\{i\}]$, generated by the identity mapping $i(z) = z$, consists of all the Koebe transforms of $i(z)$.*

In the rest of this section, unless otherwise stated, let $\mathbb{B}$ be the homogeneous unit ball of $X = \mathbb{C}^n$, that is $\mathbb{B}$ is the unit ball of a finite dimensional JB*-triple $X$. We also assume that

$$\inf\{\|z\|_e : z \in \partial\mathbb{B}\} = 1. \tag{4.1}$$

This assumption is not so strong, because for any unit ball $\mathbb{B}$ of a finite dimensional JB*-triple $X$, there exists a constant $c > 0$ such that $c\mathbb{B}$ satisfies the equality (4.1). Also, let

$$C_1 = \sup\{\|z\|_e : z \in \partial\mathbb{B}\}. \tag{4.2}$$

Taking into account the relations (4.1) and (4.2), we deduce that

$$\|z\|_X \le \|z\|_e \le C_1\|z\|_X, \quad z \in X.$$

Also, since $|\operatorname{tr}(A)| \le n\|A\|_e$ for all $A \in L(X, \mathbb{C}^n)$ by (4.1), we have

$$\operatorname{ord}\mathcal{F} \le n\|\operatorname{ord}\|_{e,1}\mathcal{F}.$$

**Theorem 4.3** *Let $\mathbb{B}$ be the unit ball of a finite dimensional JB*-triple $X$ which satisfies the condition* (4.1). *Let $\mathcal{F}$ be a linearly invariant family on $\mathbb{B}$. Then $\|\operatorname{ord}\|_{e,1}\mathcal{F} \ge 1$ holds.*

*Proof.* Let

$$\|\operatorname{ord}\|_{X,2}\mathcal{F} = \sup_{f\in\mathcal{F}} \sup_{\|z\|_X=1} \left\{\frac{1}{2}\|D^2 f(0)(z,z)\|_X\right\}.$$

Then $\|\operatorname{ord}\|_{X,2}\mathcal{F} \ge 1$ by Hamada, Honda and Kohr [26, Theorem 3.9]. Since $\|\operatorname{ord}\|_{e,1}\mathcal{F} \ge \|\operatorname{ord}\|_{X,2}\mathcal{F}$ by (4.1), we obtain the theorem. ■

Let $h_0$ be the Bergman metric on $\mathbb{B}$ at 0 and let

$$c(\mathbb{B}) = \frac{1}{2}\sup_{z,w\in\mathbb{B}} |h_0(z,w)|.$$

The following result was obtained in Hamada, Honda and Kohr [27, Theorem 4.1].

**Theorem 4.4** *Let $\mathcal{F}$ be a linearly invariant family on the unit ball $\mathbb{B}$ of a finite dimensional JB*-triple $X$. If $\operatorname{ord}\mathcal{F} = \alpha_t < \infty$, then*

$$\frac{(1-\|z\|_X)^{\alpha_t - c(\mathbb{B})}}{(1+\|z\|_X)^{\alpha_t + c(\mathbb{B})}} \le |\det Df(z)| \le \frac{(1+\|z\|_X)^{\alpha_t - c(\mathbb{B})}}{(1-\|z\|_X)^{\alpha_t + c(\mathbb{B})}}, \quad z \in \mathbb{B} \tag{4.3}$$

*for all $f \in \mathcal{F}$. If $\mathbb{B}$ is the Euclidean unit ball or the unit polydisc of $\mathbb{C}^n$, then the above estimates are sharp.*

In view of Theorem 4.4, we may prove the lower bound for $\|Df(z)\|_{X,e}$, when $f$ belongs to a L.I.F. on the unit ball of a finite dimensional JB*-triple $X$.

**Theorem 4.5** *Let $\mathbb{B}$ be the unit ball of an $n$-dimensional JB*-triple $X$ which satisfies the condition* (4.1). *Let $\mathcal{F}$ be a linearly invariant family on $\mathbb{B}$. If $\|\operatorname{ord}\|_{e,1}\mathcal{F} = \alpha < \infty$, then*

$$\frac{(1-\|z\|_X)^{\alpha - c(\mathbb{B})/n}}{(1+\|z\|_X)^{\alpha + c(\mathbb{B})/n}} \le \|Df(z)\|_{X,e} \le C_1\frac{(1+\|z\|_X)^{\alpha-1}}{(1-\|z\|_X)^{\alpha+1}}, \quad z \in \mathbb{B}, \tag{4.4}$$

*for all $f \in \mathcal{F}$, where $C_1$ is a constant defined by* (4.2).

*Proof.* Let $\operatorname{ord}\mathcal{F} = \alpha_t$. Since $|\det Df(z)| \le \|Df(z)\|_{X,e}^n$ and $\alpha_t \le n\alpha$ by the condition (4.1), the lower bound in (4.4) follows from the relation (4.3).

Next, let $\alpha_X = \|\operatorname{ord}\|_{X,1}\mathcal{F}$, where

$$\|\operatorname{ord}\|_{X,1}\mathcal{F} = \sup_{f\in\mathcal{F}} \sup_{\|w\|_X=1} \left\{ \frac{1}{2} \left\|D^2 f(0)(w,\cdot)\right\|_X \right\}.$$

Then $\alpha_X \le \alpha$ in view of the relation (4.1), and

$$\|Df(z)\|_{X,e} \le C_1 \|Df(z)\|_X \le C_1 \frac{(1+\|z\|_X)^{\alpha_X-1}}{(1-\|z\|_X)^{\alpha_X+1}} \le C_1 \frac{(1+\|z\|_X)^{\alpha-1}}{(1-\|z\|_X)^{\alpha+1}},$$

by Hamada, Honda and Kohr [26, Theorem 4.2]. ∎

Let $\mathbb{B}^n$ be the Euclidean unit ball in $\mathbb{C}^n$. Then Theorem 4.5 yields the following particular case (compare Hamada, Honda and Kohr [26] and Pfaltzgraff and Suffridge [56]). In view of Pfaltzgraff and Suffridge [56, Theorem 4.1], the upper estimate in (4.5) is sharp and the lower estimate in (4.5) is not sharp.

**Corollary 4.6** *Let $\mathcal{F}$ be a linearly invariant family on $\mathbb{B}^n$. If $\|\operatorname{ord}\|_{e,1}\mathcal{F} = \alpha < \infty$, then*

$$\frac{(1-\|z\|_e)^{\alpha-\frac{n+1}{2n}}}{(1+\|z\|_e)^{\alpha+\frac{n+1}{2n}}} \le \|Df(z)\|_e \le \frac{(1+\|z\|_e)^{\alpha-1}}{(1-\|z\|_e)^{\alpha+1}}, \quad z \in \mathbb{B}^n, \tag{4.5}$$

*for all $f \in \mathcal{F}$.*

If $\mathbb{U}^n$ is the unit polydisc in $\mathbb{C}^n$, then we obtain the following corollary, in view of Theorem 4.5 (compare Hamada and Kohr [31]).

**Corollary 4.7** *Let $\mathcal{F}$ be a linearly invariant family on $\mathbb{U}^n$. If $\|\operatorname{ord}\|_{e,1}\mathcal{F} = \alpha < \infty$, then*

$$\frac{(1-\|z\|_\infty)^{\alpha-1}}{(1+\|z\|_\infty)^{\alpha+1}} \le \|Df(z)\|_{X,e} \le \sqrt{n}\frac{(1+\|z\|_\infty)^{\alpha-1}}{(1-\|z\|_\infty)^{\alpha+1}}, \quad z \in \mathbb{U}^n, \tag{4.6}$$

*for all $f \in \mathcal{F}$, where $\|\cdot\|_\infty$ denotes the maximum norm on $\mathbb{C}^n$.*

**Open Problem 4.8** *Are the estimates in the inequalities* (4.6) *sharp?*

**Theorem 4.9** *Let $\mathbb{B}$ be the unit ball of an $n$-dimensional JB$^*$-triple $X$ which satisfies the condition* (4.1)*. Let $\mathcal{F}$ be a linearly invariant family on $\mathbb{B}$. If $\|\operatorname{ord}\|_{e,1}\mathcal{F} = \alpha < \infty$, then*

$$\frac{(1-\|z\|_X)^{(2n-1)\alpha+n-1-c(\mathbb{B})}}{(1+\|z\|_X)^{(2n-1)\alpha-n+1+c(\mathbb{B})}} \|w\|_X \le C_1^{n-1} \left\|Df(z)w\right\|_e, \quad z \in \mathbb{B},\ w \in X,$$

*for all $f \in \mathcal{F}$, where $C_1$ is a constant defined by* (4.2)*.*

*Proof.* If $A \in L(\mathbb{C}^n)$, then $\|A\|_{X,e} \ge \sqrt{\lambda_n}$, where $0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ are the eigenvalues of $A^*A$ and

$$\sqrt{\lambda_1} \le \inf\{\|Aw\|_e : \|w\|_X = 1\}.$$

Also, $|\det A| = \sqrt{\lambda_1 \cdots \lambda_n} \le \sqrt{\lambda_1}\lambda_n^{(n-1)/2}$. Since $\alpha_t \le n\alpha$, we obtain from Theorems 4.4 and 4.5 that

$$\begin{aligned}
\frac{(1-\|z\|_X)^{n\alpha-c(\mathbb{B})}}{(1+\|z\|_X)^{n\alpha+c(\mathbb{B})}} &\le \frac{(1-\|z\|_X)^{\alpha_t-c(\mathbb{B})}}{(1+\|z\|_X)^{\alpha_t+c(\mathbb{B})}} \\
&\le |\det Df(z)| = \sqrt{\lambda_1 \cdots \lambda_n} \\
&\le \sqrt{\lambda_1}\lambda_n^{(n-1)/2} \\
&\le \sqrt{\lambda_1}\left(C_1 \frac{(1+\|z\|_X)^{\alpha-1}}{(1-\|z\|_X)^{\alpha+1}}\right)^{n-1}
\end{aligned}$$

for all $z \in \mathbb{B}$. Therefore, we have

$$\frac{(1-\|z\|_X)^{(2n-1)\alpha+n-1-c(\mathbb{B})}}{(1+\|z\|_X)^{(2n-1)\alpha-n+1+c(\mathbb{B})}} \le C_1^{n-1}\sqrt{\lambda_1} \le C_1^{n-1}\|Df(z)w\|_e$$

for all $z \in \mathbb{B}$ and $w \in X$ with $\|w\|_X = 1$. This completes the proof. ∎

## 5. Distortion results for Bloch mappings

Let $\mathbb{U} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ be the unit disc in $\mathbb{C}$ and let $f : \mathbb{U} \to \mathbb{C}$ be a holomorphic function with $f'(0) = 1$. The celebrated Bloch's theorem states that $f$ maps a domain in $\mathbb{U}$ biholomorphically onto a disc with radius $r(f)$ greater than some positive absolute constant. The 'best possible' constant $\mathbf{B}$ for all such functions, that is,

$$\mathbf{B} = \inf\{r(f) : f \text{ is holomorphic on } \mathbb{U} \text{ and } f'(0) = 1\},$$

is called the Bloch constant. Bonk [3] proved the following distortion theorem.

**Theorem 5.1** *If $f : \mathbb{U} \to \mathbb{C}$ is a holomorphic function such that $f'(0) = 1$ and $\sup_{\zeta\in\mathbb{U}}(1-|\zeta|^2)|f'(\zeta)| \le 1$, then the real part $\Re f'(\zeta)$ satisfies*

$$\Re f'(\zeta) \ge \frac{1-\sqrt{3}|\zeta|}{\left(1-\dfrac{|\zeta|}{\sqrt{3}}\right)^3}, \quad |\zeta| \le \frac{1}{\sqrt{3}}.$$

The above distortion theorem implies readily a result of Ahlfors [1] that the Bloch constant $\mathbf{B}$ is greater than $\dfrac{\sqrt{3}}{4}$. This lower bound was further improved by Bonk [3] to $\mathbf{B} > \dfrac{\sqrt{3}}{4} + 10^{-14}$, and by Chen and Gauthier [7] to $\mathbf{B} \ge \dfrac{\sqrt{3}}{4} + 2 \times 10^{-4}$.

The notion of a $\mathbb{C}^n$-valued Bloch mapping on a finite dimensional bounded symmetric domain, under the name of *normal mapping of finite order*, was first introduced by Hahn [21]. Several equivalent definitions for complex-valued Bloch functions on a finite dimensional bounded homogeneous domain have been given by Timoney [62]. $\mathbb{C}^n$-valued Bloch mappings on the Euclidean ball of $\mathbb{C}^n$ have also been studied in Liu [49]. The following definition for a Bloch mapping from a finite dimensional bounded symmetric domain to $\mathbb{C}^n$ given by Hamada [23] is a direct extension of the one in Timoney [62, Theorem 3.4 (4)] and Liu [49].

**Definition 5.2** *Let $\mathbb{B}_X$ be the unit ball of a finite dimensional JB*-triple $X$. A mapping $f \in H(\mathbb{B}_X, \mathbb{C}^n)$ is called a Bloch mapping if the family*

$$F_f = \{f \circ \varphi - f(\varphi(0)) : \varphi \in \mathrm{Aut}(\mathbb{B}_X)\}$$

*is normal, that is, every sequence in $F_f$ contains a subsequence converging uniformly on compact subsets of $\mathbb{B}_X$.*

Equivalently, $f \in H(\mathbb{B}_X, \mathbb{C}^n)$ is a Bloch mapping if

$$\|f\|_{\mathcal{B}} = \sup \{\|D(f \circ \varphi)(0)\|_{X,e} : \varphi \in \mathrm{Aut}(\mathbb{B}_X)\} < \infty$$

(cf. Liu [49], Timoney [62]), where $\|f\|_{\mathcal{B}}$ is called the *Bloch semi-norm* of $f$.

For $1 \leq K \leq +\infty$, we will denote by $\beta(B_X, \mathbb{C}^n, K)$ the set of Bloch mappings $f \in H(\mathbb{B}_X, \mathbb{C}^n)$ with $\|f\|_{\mathcal{B}} \leq K$.

We note that, in the above definition of a $\mathbb{C}^n$-valued Bloch mapping, we do not require that the domain $\mathbb{B}_X$ has the same dimension $n$, although this is the case in the following results. We recall that an $n$-dimensional JB*-triple $X$ is the complex space $(\mathbb{C}^n, \|\cdot\|_X)$ equipped with the Carathéodory norm $\|\cdot\|_X$.

**Definition 5.3** *Let $\mathbb{B}_X$ be the unit ball of an $n$-dimensional JB*-triple $X$. We define the prenorm $\|f\|_0$ of $f \in H(\mathbb{B}_X, \mathbb{C}^n)$ by*

$$\|f\|_0 = \sup \left\{(1 - \|z\|^2)^{c(\mathbb{B}_X)/n} |\det Df(z)|^{1/n} : z \in \mathbb{B}_X \right\}.$$

Bonk's distortion theorem has been extended by Liu [49, Theorem 7] to the family $H_{\mathrm{loc}}(\mathbb{B}^n, \mathbb{C}^n)$ of $\mathbb{C}^n$-valued locally biholomorphic mappings on the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, as follows.

**Theorem 5.4** *If $f \in H_{\mathrm{loc}}(\mathbb{B}^n, \mathbb{C}^n)$, $\|f\|_0 = 1$ and $\det Df(0) = 1$, then*

$$|\det Df(z)| \geq \Re \det Df(z) \geq \frac{\exp\left(\dfrac{-(n+1)\|z\|}{1 - \|z\|}\right)}{(1 - \|z\|)^{n+1}}, \quad z \in \mathbb{B}^n.$$

*This inequality is sharp.*

Bloch's theorem fails in dimension 2. Nevertheless, one can define the Bloch constant for various families of Bloch mappings in higher dimensions. Using the above distortion theorem, lower and upper bounds for such a Bloch constant for $\mathbb{B}^n$ were obtained in Liu [49]. For the class $H_{\mathrm{loc}}(\mathbb{U}^n, \mathbb{C}^n)$ of locally biholomorphic mappings on the unit polydisc $\mathbb{U}^n$ in $\mathbb{C}^n$, the following distortion theorem has been shown by Wang and Liu [66, Theorem 3.2].

**Theorem 5.5** *If $f \in H_{\mathrm{loc}}(\mathbb{U}^n, \mathbb{C}^n)$, $\|f\|_0 = 1$ and $\det Df(0) = 1$, then*

$$|\det Df(z)| \geq \Re \det Df(z) \geq \frac{\exp\left(\dfrac{-2n\|z\|}{1 - \|z\|}\right)}{(1 - \|z\|)^{2n}}, \quad z \in \mathbb{U}^n.$$

*This inequality is sharp.*

This theorem was also used in Wang and Liu [66] to derive a lower bound of the Bloch constant for classes of locally biholomorphic Bloch mappings on $\mathbb{U}^n$.

Both the Euclidean unit ball and the unit polydisc in $\mathbb{C}^n$ are examples of bounded symmetric domains in $\mathbb{C}^n$. The following natural questions arise.

**Question 5.6** *Can we explain the difference of the exponents in the distortion bounds in Theorems* 5.4 *and* 5.5*?*

**Question 5.7** *Can we extend Bonk's distortion theorem to other bounded symmetric domains in* $\mathbb{C}^n$*?*

We give an affirmative answer to both questions in this talk and as an application, we derive a lower bound of the Bloch constant for various classes of locally biholomorphic Bloch mappings on a finite dimensional bounded symmetric domain.

**Theorem 5.8** *Let* $\mathbb{B}_X$ *be the unit ball of an* $n$*-dimensional JB*-triple* $X$*. Let* $\alpha \in (0,1]$ *and let* $m(\alpha)$ *be the unique root of the equation*

$$e^{-c(\mathbb{B}_X)x}(1+x)^{c(\mathbb{B}_X)} = \alpha \tag{5.1}$$

*in the interval* $[0,+\infty)$*. If* $f \in H_{\text{loc}}(\mathbb{B}_X, \mathbb{C}^n)$*,* $\|f\|_0 = 1$ *and* $\det Df(0) = \alpha$*, then we have*

(*i*)

$$|\det Df(z)| \geq \frac{\alpha}{(1-\|z\|)^{2c(\mathbb{B}_X)}} \exp\left\{(1+m(\alpha))\frac{-2c(\mathbb{B}_X)\|z\|}{1-\|z\|}\right\} \tag{5.2}$$

*for* $z \in \mathbb{B}_X$*;*

(*ii*)

$$|\det Df(z)| \leq \frac{\alpha}{(1+\|z\|)^{2c(\mathbb{B}_X)}} \exp\left\{(1+m(\alpha))\frac{2c(\mathbb{B}_X)\|z\|}{1+\|z\|}\right\} \tag{5.3}$$

*for* $\|z\| \leq \dfrac{m(\alpha)}{2+m(\alpha)}$.

*The estimates in* (5.2) *and* (5.3) *are sharp.*

A special case of Theorem 5.8 asserts that

$$|\det Df(z)| \geq \frac{1}{(1-\|z\|)^{2c(\mathbb{B}_X)}} \exp\left\{\frac{-2c(\mathbb{B}_X)\|z\|}{1-\|z\|}\right\}$$

for $f \in H_{\text{loc}}(\mathbb{B}_X, \mathbb{C}^n)$, $\|f\|_0 = 1$ and $\det Df(0) = 1$. This generalizes Theorems 5.4 and 5.5, and also explains the difference of the exponents in the first question.

Our results also generalize simultaneously other results on Bonk's distortion theorem for locally univalent Bloch functions in one complex variable in Bonk, Minda and Yanagihara [4], Liu and Minda [50], and those for locally biholomorphic Bloch mappings in several complex variables in Wang [64]. We refer to Chu, Hamada, Honda and Kohr [10], Hamada, Honda and Kohr [26, 27, 28] for other distortion theorems for normalized locally biholomorphic mappings on unit balls of finite dimensional JB*-triples.

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