特別講演

Painlevé functions, Fredholm determinants and combinatorics

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Abstract

We are going to explain explicit construction of general solutions to isomonodromy equations, with the main focus on the Painlevé VI equation. We will start by deriving Fredholm determinant representation of the Painlevé VI tau function. The corresponding integral operator acts in the direct sum of two copies of $L^2(S^1)$. Its kernel is expressed in terms of hypergeometric fundamental solutions of two auxiliary 3-point Fuchsian systems whose monodromy is determined by monodromy of the associated linear problem via a decomposition of $\mathbb{CP}^1\backslash\{4\text{ points}\}$ into two pairs of pants. In the Fourier basis, this kernel is given by an infinite Cauchy matrix. It will be shown that the principal minor expansion of the Fredholm determinant yields a combinatorial series representation for the general solution to Painlevé VI in the form of a sum over pairs of Young diagrams.

1 Introduction

The goal of the talk is to explain explicit construction of general solutions of certain classes of monodromy preserving deformation equations. We are going to focus on the paradigmatic example of the sixth Painlevé equation.

The Painlevé VI equation describes isomonodromic deformations of the Fuchsian system

$$\begin{aligned} &\partial_z\Phi = \Phi A(z), \\ &A(z) = \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}, \qquad A_\nu \in \mathfrak{sl}_2, \end{aligned} \tag{1.1}$$

with 4 regular singular points given by the simple poles of the 1-form $A(z)\,dz$ on $\mathbb{CP}^1$. Three of these points can be fixed as above at $0, 1, \infty$ by a Möbius transformation. The position of the 4th singularity $z = t$ (anharmonic ratio) will play below the role of time variable. The fundamental matrix $\Phi(z)$ is holomorphic on the universal cover of $\mathbb{C}\backslash\{0, t, 1\}$. Its analytic continuation induces a monodromy representation of the fundamental group, $\rho : \pi_1\left(\mathbb{CP}^1\backslash\{4\text{ points}\}\right) \to \mathrm{SL}(2, \mathbb{C})$. This defines the Riemann-Hilbert map $\mathcal{RH}$ from the space $\mathcal{P}$ of parameters of the Fuchsian system to the space $\mathcal{M}$ of the relevant monodromy data.

The requirement of conservation of monodromy upon variation of t yields a (Schlesinger) system of matrix ODEs

$$\begin{cases} \dot{A}_0 = \dfrac{[A_0, A_t]}{t}, \\ \dot{A}_1 = \dfrac{[A_1, A_t]}{t-1}, \\ A_\infty := -A_0 - A_t - A_1 = \mathrm{const}, \end{cases}$$

where the dots denote derivatives with respect to t. The latter system can be equivalently recast into a single scalar 2nd order ODE

$$\left(t(t-1)\ddot{\zeta}\right)^2 = -2\det\begin{pmatrix} 2\theta_0^2 & t\dot{\zeta}-\zeta & \dot{\zeta}+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 \\ t\dot{\zeta}-\zeta & 2\theta_t^2 & (t-1)\dot{\zeta}-\zeta \\ \dot{\zeta}+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 & (t-1)\dot{\zeta}-\zeta & 2\theta_1^2 \end{pmatrix}, \tag{1.2}$$

for the function $\zeta(t) = (t-1)\operatorname{Tr} A_0A_t + t\operatorname{Tr} A_1A_t$. The latter equation is nothing but Painlevé VI in a disguised form, $\zeta(t)$ being closely connected to its non-autonomous hamiltonian. Four extra parameters

$\theta_{0,t,1,\infty}$ therein correspond to the conserved eigenvalues $\pm\theta_\nu$ of A_ν; they also represent exponents of local monodromy. The space $\mathcal{M}$ of monodromy data is 6-dimensional. Two remaining parameters encode Painlevé VI initial conditions. Finding the general solution of Painlevé VI thereby amounts to constructing explicit inverse $\mathcal{RH}^{-1}$ of the Riemann-Hilbert map.

By 2012, several families of explicit solutions of Painlevé VI had been discovered. In all of them, one has to impose constraints on monodromy — namely, on the parameters of the equation and, in most cases, on the initial conditions:

- *Riccati family.* These PVI solutions correspond, up to Bäcklund transformations, to reducible monodromy (1 constraint on θ's, 1-parameter initial conditions).
- *Picard family.* Here the parameters are fixed and are Bäcklund-equivalent to $\theta_0 = \theta_t = \theta_1 = \theta_\infty = \frac{1}{4}$; no constraints on the initial conditions.
- *Algebraic solutions.* There exists a finite number of equivalence classes of algebraic solutions, most of which correspond to isolated points of $\mathcal{M}$; there are also continuous families depending on 1 or 2 complex parameters.
- ${}_2F_1$ *kernel determinant.* The Fredholm determinant of the continuous hypergeometric kernel is related to Painlevé VI [BD] in a way similar to the familiar expression of the Airy kernel determinant (Tracy-Widom distribution) in terms of the Hastings-McLeod solution of Painlevé II. Here $\theta_t = 0$ and the initial conditions form a 1-parameter family.

Looking for the general Painlevé VI transcendent, the last family of solutions may seem the most promising candidate for generalizations. Yet the answer has been first found in [GIL12] in a different form of combinatorial series involving a double sum over the set $\mathbb{Y}$ of Young diagrams. To formulate the result, introduce the Painlevé VI tau function $\tau_{\mathrm{VI}}(t)$ by [JMU]

$$\zeta(t) = t(t-1)\,\partial_t \ln \tau_{\mathrm{VI}}(t)\,. \tag{1.3}$$

We also need to set up the notation for Young diagrams. Given $\lambda \in \mathbb{Y}$ drawn in English convention, λ' denotes the conjugate diagram, λ_i and λ'_j are the number of boxes in the ith row and jth column of λ, and $|\lambda|$ stands for the total number of boxes in λ. Let (i,j) be the box in the ith row and jth column of $\lambda \in \mathbb{Y}$, see Fig. 1. Its hook length is defined as $h_\lambda(i,j) = \lambda_i + \lambda'_j - i - j + 1$.

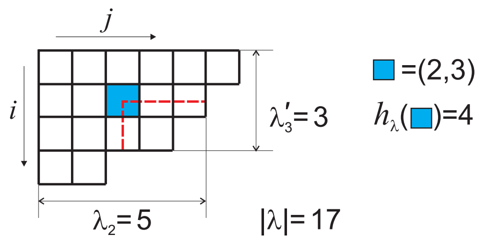

Figure 1: Notation for Young diagrams.

Conjecture [GIL12]. *General solution of the Painlevé VI equation (1.2) can be written as*

$$\tau_{\mathrm{VI}}(t) = \mathrm{const}\cdot\sum_{n\in\mathbb{Z}} e^{in\eta}\mathcal{B}\big(\vec{\theta};\sigma+n;t\big), \tag{1.4a}$$

where $\mathcal{B}\big(\vec{\theta},\sigma;t\big)$ *is a double sum over Young diagrams,*

$$\mathcal{B}\big(\vec{\theta},\sigma;t\big) = \mathcal{N}^{\theta_1}_{\theta_\infty,\sigma}\mathcal{N}^{\theta_t}_{\sigma,\theta_0}\, t^{\sigma^2-\theta_0^2-\theta_t^2}(1-t)^{2\theta_t\theta_1}\sum_{\lambda,\mu\in\mathbb{Y}}\mathcal{B}_{\lambda,\mu}\big(\vec{\theta},\sigma\big)\,t^{|\lambda|+|\mu|}, \tag{1.4b}$$

$$\mathcal{B}_{\lambda,\mu}(\vec{\theta},\sigma) = \prod_{(i,j)\in\lambda} \frac{\left((\theta_t+\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1+\sigma+i-j)^2-\theta_\infty^2\right)}{h_\lambda^2(i,j)\left(\lambda'_j-i+\mu_i-j+1+2\sigma\right)^2}\times$$
$$\times \prod_{(i,j)\in\mu} \frac{\left((\theta_t-\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1-\sigma+i-j)^2-\theta_\infty^2\right)}{h_\mu^2(i,j)\left(\mu'_j-i+\lambda_i-j+1-2\sigma\right)^2}, \tag{1.4c}$$

$$\mathcal{N}^{\theta_2}_{\theta_3,\theta_1} = \frac{\prod_{\epsilon=\pm} G\left(1+\theta_3+\epsilon(\theta_1+\theta_2)\right) G\left(1-\theta_3+\epsilon(\theta_1-\theta_2)\right)}{G(1-2\theta_1)G(1-2\theta_2)G(1+2\theta_3)}. \tag{1.4d}$$

Here $\sigma \notin \mathbb{Z}/2$, η are two arbitrary complex parameters representing the initial conditions, and $G(z)$ denotes the Barnes G-function; its only property relevant for the above is the recurrence relation $G(z+1) = \Gamma(z)\, G(z)$.

The above gives a series representation for $\tau_{\rm VI}(t)$ around $t=0$. Similar expansions can be written around the two other branch points $t=1,\infty$. The leading terms of these series reproduce Jimbo's asymptotic formula for Painlevé VI [Jim]. The parameters σ,η are Fenchel-Nielsen type coordinates on the space $\mathcal{M}$ of monodromy data.

The nontrivial part of the claim of [GIL12] is the equation (1.4a) together with the identification of the function $\mathcal{B}(\vec{\theta},\sigma;t)$ with the four-point $c=1$ conformal block of the Virasoro algebra. Combinatorial representation of this function is a consequence of the correspondence [AGT] between 2D conformal field theories and 4D supersymmetric gauge theories. By now, the former statement is well understood in the CFT framework [ILT, BSh1], and has been extended to the Garnier system [ILT] as well as to some irregular [GIL13, Nag1, Nag2] and q-difference [BSh2, JNS] isomonodromic problems.

The question that we want to address here is how to prove combinatorial formulas of the above conjecture rigorously and directly, i.e. bypassing the use of CFT arguments and AGT correspondence. The plan of the proof is as follows:

1. First we are going to transform the 4-point Fuchsian system (1.1) into a Riemann-Hilbert problem on a circle, where the appropriate jump matrix involves solutions of two auxiliary 3-point Fuchsian systems.

2. Next we will establish a Fredholm determinant representation of $\tau_{\rm VI}(t)$. The corresponding integral operator acts in the direct sum of two copies of $L^2(S^1)$, and its kernel is expressed in terms of hypergeometric 3-point solutions.

3. Expanding the Fredholm determinant into a sum of principal minors in the basis of Fourier modes leads to combinatorial series (1.4).

The presented approach has been developed in [GL16], with important further generalizations and simplifications discussed in [GL17, CGL].

2 Riemann-Hilbert problem

Consider generic situation where $A_{0,t,1,\infty}$ are diagonalizable. Fix the diagonalizations $A_\nu = G_\nu^{-1}\Theta_\nu G_\nu$ with $\Theta_\nu = \operatorname{diag}\{\theta_\nu,-\theta_\nu\}$ and assume that all $\theta_\nu \notin \mathbb{Z}/2$. Then there exist unique fundamental matrix solutions $\Phi^{(\nu)}(z)$ of (1.1), holomorphic on the universal covering of $\mathbb{C}\backslash\{0,t,1\}$ and such that

$$\Phi^{(\nu)}(z) = \begin{cases} (\nu - z)^{\Theta_\nu} G^{(\nu)}(z), & \text{for } \nu = 0,t,1, \\ (-z)^{-\Theta_\infty} G^{(\infty)}(z), & \text{for } \nu=\infty, \end{cases}$$

where $G^{(\nu)}(z)$ is holomorphic and invertible in a finite open disk around $z=\nu$ and satisfies the normalization condition $G^{(\nu)}(\nu) = G_\nu$.

Further assume for notational simplicity that $t\in(0,1)$. The canonical solutions $\Phi^{(0,\infty)}(z)$ analytically continue to holomorphic functions on $\mathbb{CP}^1\backslash\mathbb{R}_{\geq 0}$. Similarly, $\Phi^{(t)}(z)$ and $\Phi^{(1)}(z)$ are naturally defined on $\mathbb{CP}^1\backslash((-\infty,0]\cup[t,\infty))$ and $\mathbb{CP}^1\backslash((-\infty,t]\cup[1,\infty))$, respectively. Take an arbitrary fundamental solution $\Phi(z)$, defined on $\mathbb{CP}^1\backslash\mathbb{R}_{\geq 0}$. The connection matrices $C_{\nu,\epsilon}=\Phi(z)\,\Phi^{(\nu)}(z)^{-1}$, with $\epsilon=\operatorname{sgn}\Im z$, are independent of z. They satisfy the compatibility conditions

$$
\begin{gathered}
C_{0,+}=C_{0,-}, \qquad C_{\infty,+}=C_{\infty,-},\\
M_0=C_{0,-}e^{2\pi i\Theta_0}C_{0,+}^{-1}=C_{t,-}C_{t,+}^{-1}, \qquad M_\infty^{-1}=C_{1,-}e^{2\pi i\Theta_1}C_{1,+}^{-1}=C_{\infty,-}e^{-2\pi i\Theta_\infty}C_{\infty,+}^{-1},\\
M_0M_t=(M_1M_\infty)^{-1}=C_{t,-}e^{2\pi i\Theta_t}C_{t,+}^{-1}=C_{1,-}C_{1,+}^{-1}.
\end{gathered}
$$

where M_ν denotes anticlockwise monodromy matrix of $\Phi(z)$ around the Fuchsian singular point $\nu\in\{0,t,1,\infty\}$. The (conjugacy class of) connection matrices $\{C_{\nu,\pm}\}$ and exponents $\{\Theta_\nu\}$ of local monodromy constitute the monodromy data for the 4-point Fuchsian system (1.1).

Let $\Gamma\subset\mathbb{CP}^1$ be an oriented contour consisting of a finite number of smooth curves intersecting transversally, and let $J(z)$ be an $\mathrm{SL}(2,\mathbb{C})$-valued function on Γ. We will assign to the pair (Γ,J) two Riemann-Hilbert problems (RHPs). They ask to find functions $\Psi(z)$, $\bar\Psi(z)$ holomorphic on $\mathbb{CP}^1\backslash\Gamma$ such that their boundary values on the positive and negative side of Γ satisfy

$$\text{direct RHP}: \quad J(z)=\Psi_-(z)^{-1}\Psi_+(z), \tag{2.1a}$$

$$\text{dual RHP}: \quad J(z)=\bar\Psi_+(z)\,\bar\Psi_-(z)^{-1}. \tag{2.1b}$$

We are going to transform (1.1) into a RHP on a circle. This is achieved in several steps:

1. Start with the contour $\tilde\Gamma$ shown in Fig. 2a by solid black curves. Denote by D_ν the disk bounded by the circle isolating the point $z=\nu$ and define

$$\tilde\Psi(z)=\begin{cases} G^{(\nu)}(z), & z\in D_\nu,\\ \Phi(z), & z\notin\mathbb{R}_{\geq0}\cup\bar D_0\cup\bar D_t\cup\bar D_1\cup\bar D_\infty.\end{cases}$$

   Comparing with (2.1b), we see that the matrix function $\tilde\Psi(z)$ solves a dual RHP set on $\tilde\Gamma$ with the jumps indicated in Fig. 2a.

2. Next cancel the constant jump $(M_0M_t)^{-1}$ on the real segment cut out by the dashed red circles $\mathcal{C}_{\text{out,in}}$. To this end, let us write $M_0M_t=e^{2\pi i\mathfrak{S}}$ with $\mathfrak{S}=\operatorname{diag}\{\sigma,-\sigma\}$ (in the generic situation we may choose to work in the basis where M_0M_t is diagonal). Denote by $\hat{\mathcal{A}}$ the open annulus bounded by $\mathcal{C}_{\text{out,in}}$ and set

$$\hat\Psi(z)=\begin{cases}(-z)^{-\mathfrak{S}}\,\tilde\Psi(z), & z\in\hat{\mathcal{A}},\\ \tilde\Psi(z), & z\notin\bar{\hat{\mathcal{A}}}.\end{cases}$$

   The dual RHP for $\hat\Psi(z)$ is set on the contour $\hat\Gamma$ indicated in Fig. 2b by solid black lines. The jump matrices associated to $\mathcal{C}_{\text{out}}$ and $\mathcal{C}_{\text{in}}$ are $(-z)^{-\mathfrak{S}}$; on the rest of the contour the jumps are the same as for $\tilde\Psi(z)$.

3. The contour $\hat\Gamma$ has two connected components, Γ_+ and Γ_-, containing respectively $\mathcal{C}_{\text{out}}$ and $\mathcal{C}_{\text{in}}$. The RHPs obtained by restricting the initial contour to Γ_+ or Γ_- while keeping the same jumps are then generically solvable. Their solutions are related to fundamental matrices of 3-point Fuchsian systems whose singular poins are $0,t,\infty$ and $0,1,\infty$. Let us suggestively denote these solutions by $\Psi_-(z)$ and $\Psi_+(z)$. The subscript reminds that these functions are analytic outside and inside $\mathcal{C}$, respectively.

   Consider an auxiliary circle $\mathcal{C}$ inside $\hat{\mathcal{A}}$, indicated by dashed red line in Fig. 2b, and define

$$\bar\Psi(z)=\begin{cases}\Psi_+(z)^{-1}\hat\Psi(z), & \text{outside } \mathcal{C},\\ \Psi_-(z)^{-1}\hat\Psi(z), & \text{inside } \mathcal{C}.\end{cases}$$

The matrix function $\bar{\Psi}(z)$ has no jumps except on $\mathcal{C}$. The jump of the relevant *dual* RHP is written in the form of *direct* factorization,

$$J(z) = \Psi_-(z)^{-1}\Psi_+(z), \tag{2.2}$$

cf (2.1a). The problem of solving the 4-point Fuchsian system with a prescribed monodromy is now converted into a RHP on a single circle $\mathcal{C}$ (Fig. 2c) and the jump matrix expressed in terms of 3-point solutions. The latter can be found explicitly and expressed in terms of Gauss hypergeometric functions.

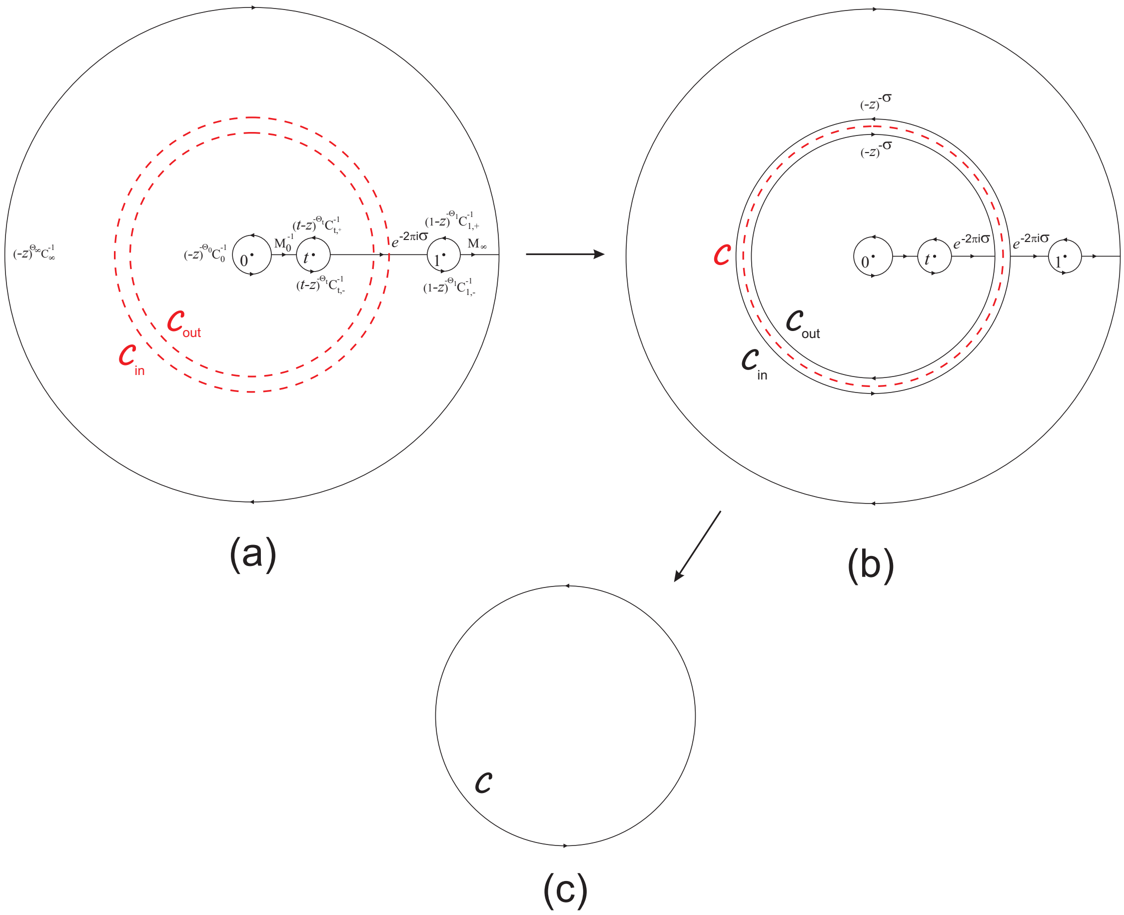

Figure 2: Transformation of contours of RHPs for (a) $\tilde{\Psi}$ (b) $\hat{\Psi}$ (c) $\bar{\Psi}$.

3 Tau function as Fredholm determinant

Introduce the Hilbert space $H = L^2(\mathcal{C}, \mathbb{C}^2)$. Its elements will be seen as column vector functions. This space can be decomposed as $H = H_+ \oplus H_-$, where the functions from H_+ (H_-) continue analytically inside $\mathcal{C}$ (resp. outside $\mathcal{C}$ and vanish at ∞). We denote by $\Pi_\pm$ the projections on $H_\pm$ along $H_\mp$.

Definition 3.1. *The tau function of the RHPs defined by* $(\mathcal{C}, J)$ *is defined as Fredholm determinant*

$$\tau[J] = \det_{H_+}\left(\Pi_+ J^{-1}\Pi_+ J\,\Pi_+\right). \tag{3.1}$$

Remark 3.2. Expand J inside $\mathcal{A}$ into Laurent series, $J(z) = \sum_{k\in\mathbb{Z}} J_k z^k$, and consider block Toeplitz matrix

$$T_K[J] = \begin{pmatrix} J_0 & J_{-1} & J_{-2} & \ldots & J_{-K+1} \\ J_1 & J_0 & J_{-1} & \ldots & J_{-K+2} \\ J_2 & J_1 & J_0 & \ldots & J_{-K+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J_{K-1} & J_{K-2} & J_{K-3} & \ldots & J_0 \end{pmatrix}.$$

A celebrated theorem of Widom [W2] states that $\lim_{K\to\infty} \det T_K[J] = \tau[J]$. The relevant Fredholm determinant is called the (Szegö-)Widom constant in this context.

Suppose that $J(z)$ admits a direct factorization (2.1a). Define two Cauchy-Plemelj operators $\mathsf{a} = \Psi_+\Pi_+\Psi_+^{-1}\big|_{H_-}$, $\mathsf{d} = \Psi_-\Pi_-\Psi_-^{-1}\big|_{H_+}$. They can be explicitly written as integral operators

$$(\mathsf{a}f)(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \mathsf{a}(z,z')\, f(z')\, dz', \qquad (\mathsf{d}f)(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \mathsf{d}(z,z')\, f(z')\, dz',$$

where

$$\mathsf{a}(z,z') = \frac{\mathbf{1} - \Psi_+(z)\,\Psi_+(z')^{-1}}{z-z'}, \qquad \mathsf{d}(z,z') = \frac{\Psi_-(z)\,\Psi_-(z')^{-1} - \mathbf{1}}{z-z'}. \tag{3.2}$$

The integral kernels $\mathsf{a}(z,z')$ and $\mathsf{d}(z,z')$ have integrable form, are not singular on the diagonal $z = z'$ and extend to analytic functions on $\mathcal{A}\times\mathcal{A}$.

Proposition 3.3. *If the direct RHP (2.1a) is solvable, then $\tau[J]$ can be alternatively rewritten as*

$$\tau[J] = \det\nolimits_H(\mathbf{1}+K), \qquad K = \begin{pmatrix} 0 & \mathsf{a} \\ \mathsf{d} & 0 \end{pmatrix} \in \operatorname{End}(H_+\oplus H_-), \tag{3.3}$$

where integral operators a and d have block integrable kernels defined by (3.2).

The relation between $\tau[J]$ and the Painlevé VI tau function $\tau_{\mathrm{VI}}(t)$ can be obtained using another result of Widom [W1]. The differentiation formula below may be considered a precursor of the Jimbo-Miwa-Ueno definition of the isomonodromic tau function.

Theorem 3.4. *Consider a smooth family of $\mathrm{SL}(2,\mathbb{C})$-loops $(z,t)\mapsto J(z,t)$ depending on additional parameter t and admitting both factorizations (2.1). Then*

$$\partial_t \ln\tau[J] = \frac{1}{2\pi i}\oint_{\mathcal{C}} \operatorname{Tr}\left\{J^{-1}\partial_t J\left[\partial_z\bar{\Psi}_-\,\bar{\Psi}_-^{-1} + \Psi_+^{-1}\,\partial_z\Psi_+\right]\right\} dz. \tag{3.4}$$

In the situation we are interested in, the matrices $\bar{\Psi}_\pm(z)$, $\Psi_\pm(z)$ are expressed in terms of the fundamental solutions of 4-point and 3-point Fuchsian systems, which allows to find their z- and t-derivatives. This ultimately reduces the integration in (3.4) to residue calculation and yields our first important result:

Corollary 3.5. *We have the following identification:*

$$\tau_{\mathrm{VI}}(t) = \mathrm{const}\cdot t^{\sigma^2-\theta_0^2-\theta_t^2}\,\tau[J], \tag{3.5}$$

with jump matrix J defined by (2.2). Proposition 3.3 thereby provides an explicit Fredholm determinant representation for $\tau_{\mathrm{VI}}(t)$, the relevant integral operator acting in $H = L^2(\mathcal{C})\oplus L^2(\mathcal{C})$.

4 Combinatorial series

Let $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$ be the half-integer lattice, $\mathbb{Z}'_{\pm} = \mathbb{Z}'_{\gtrless 0}$, and let $\mathrm{Conf}\,(\mathbb{Z}') = \{0,1\}^{\mathbb{Z}'}$ be the set of all finite subsets of $\mathbb{Z}'$. The elements $X \subset \mathrm{Conf}\,(\mathbb{Z}')$ determine the positions of particles $\mathsf{p}_X := X \cap \mathbb{Z}'_+$ and holes $\mathsf{h}_X := X \cap \mathbb{Z}'_-$ thereby defining point configurations on $\mathbb{Z}'$. A configuration X may be alternatively represented by

- A Maya diagram m_X obtained by drawing filled circles at sites $\left(\mathbb{Z}'_+ \setminus \mathsf{p}_X\right) \cup \mathsf{h}_X$ and empty circles at $\mathsf{p}_X \cup \left(\mathbb{Z}'_- \setminus \mathsf{h}_X\right)$, see Fig. 3. The charge of m_X is defined as $Q_X = |\mathsf{p}_X| - |\mathsf{h}_X|$. The set of all Maya diagrams will be denoted by $\mathbb{M}$.
- A charged partition/Young diagram $(\lambda_X, Q_X) \in \mathbb{Y} \times \mathbb{Z}$. The Maya diagram corresponding to (λ_X, Q_X) can be described by the positions of empty circles, given by $\left\{(\lambda_X)_k - k + \frac{1}{2} + Q_X\right\}_{k=1}^{\infty}$, cf Fig. 3.

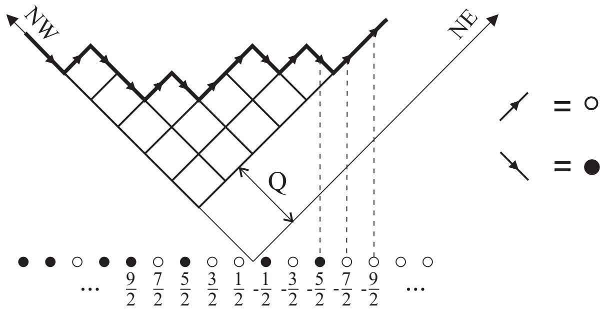

Figure 3: Correspondence between Maya and charged Young diagrams. The positions of particles and holes are $\mathsf{p}_X = \left\{\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, \frac{13}{2}\right\}$ and $\mathsf{h}_X = \left\{-\frac{5}{2}, -\frac{1}{2}\right\}$. The charge $\mathsf{Q}_X = 2$ corresponds to the signed distance between the north-east axis and appropriate boundary of the profile of λ_X.

Let $K \in \mathbb{C}^{\mathfrak{X}\times\mathfrak{X}}$ be a matrix indexed by a discrete set $\mathfrak{X}$; the latter can be infinite, in which case K is required to be a trace class operator on $\ell^2(\mathfrak{X})$. The determinant $\det(\mathbf{1}+K)$ can be expressed as the sum of principal minors enumerated by all possible subsets of $\mathfrak{X}$:

$$\det(\mathbf{1}+K) = \sum_{\mathfrak{Y}\in\{0,1\}^{\mathfrak{X}}} \det K_{\mathfrak{Y}},$$

i.e. $K_{\mathfrak{Y}}$ is the restriction of K to rows and columns labeled by elements of $\mathfrak{Y}$.

We now apply this formula to the determinant (3.3). Rewrite the integral operators a and d in the Fourier basis. Their kernels (3.2) may be expressed as

$$\mathsf{a}(z,z') = \sum_{p,q\in\mathbb{Z}'_+} \mathsf{a}_{-q}^{\;p} z^{-\frac{1}{2}+p} z'^{-\frac{1}{2}+q}, \qquad \mathsf{d}(z,z') = \sum_{p,q\in\mathbb{Z}'_+} \mathsf{d}^{-q}_{\;\;p} z^{-\frac{1}{2}-q} z'^{-\frac{1}{2}-p}, \tag{4.1}$$

where the coefficients $\mathsf{a}_{-q}^{\;p}, \mathsf{d}^{-q}_{\;\;p} \in \mathrm{Mat}_{2\times2}(\mathbb{C})$ are themselves matrices whose elements we write as $\mathsf{a}_{-q;\beta}^{\;p;\alpha}, \mathsf{d}^{-q;\alpha}_{\;\;p;\beta}$. The "color" indices $\alpha, \beta \in \{1,2\}$ correspond to matrix structure of the RHP defined by the loop J. The principal minors of K in (3.3) are therefore labeled by pairs of Maya diagrams

$$\mathbf{m} = (\mathsf{m}_1, \mathsf{m}_2) = (\mathbf{p}, \mathbf{h}) \in \mathbb{M}^2,$$
$$\mathbf{p} = \mathsf{p}_1 \sqcup \mathsf{p}_2, \qquad \mathbf{h} = \mathsf{h}_1 \sqcup \mathsf{h}_2.$$

Here $\mathsf{p}_\alpha \in \{0,1\}^{\mathbb{Z}'_+}$, $\mathsf{h}_\alpha \in \{0,1\}^{\mathbb{Z}'_-}$ denote the positions of particles and holes of color $\alpha \in \{1,2\}$. The minors with $|\mathbf{p}| \neq |\mathbf{h}|$ clearly vanish, cf Fig. 4. We may thus restrict the summation to pairs of Maya

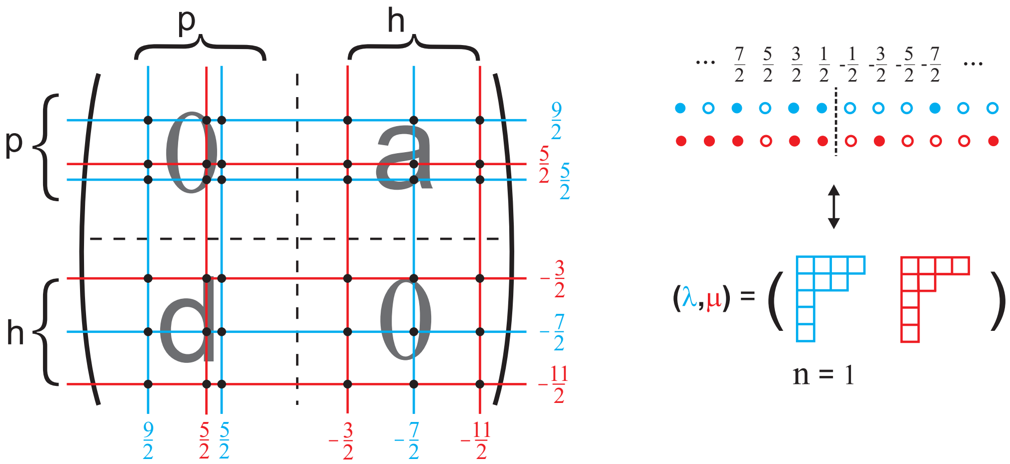

Figure 4: Example of labeling of principal minors.

diagrams of zero total charge,

$$\tau[J] = \sum_{\mathbf{m}\in\mathbb{M}^2:\,|\mathbf{p}|=|\mathbf{h}|} Z^{[+]}_{\mathbf{m}} Z^{[-]}_{\mathbf{m}}, \qquad Z^{[+]}_{\mathbf{m}} = \det \mathsf{a}^{\,\mathbf{p}}_{\,\mathbf{h}}, \qquad Z^{[-]}_{\mathbf{m}} = (-1)^{|\mathbf{p}|} \det \mathsf{d}^{\,\mathbf{h}}_{\,\mathbf{p}}. \tag{4.2}$$

The matrices $\mathsf{a}^{\,\mathbf{p}}_{\,\mathbf{h}}, \mathsf{d}^{\,\mathbf{h}}_{\,\mathbf{p}} \in \mathrm{Mat}_{|\mathbf{p}|\times|\mathbf{p}|}(\mathbb{C})$ correspond to the upper-right and lower-left block in the principal minor in Fig. 4. Using the identification of Maya diagrams and charged partitions described above, the individual contributions to (4.2) may also be labeled by a pair of Young diagrams $(\lambda,\mu)\in\mathbb{Y}^2$ and an integer $n = Q_\lambda = -Q_\mu$ defining the charges assigned to λ and μ. Adapting the notation, the combinatorial expansion (4.2) may then be written as

$$\tau[J] = \sum_{n\in\mathbb{Z}} \sum_{\lambda,\mu\in\mathbb{Y}} Z^{[+]}_{\lambda,\mu,n} Z^{[-]}_{\lambda,\mu,n}. \tag{4.3}$$

Combining the last result with Corollary 3.5, the combinatorial structure of (1.4a)–(1.4b) becomes manifest. To finish the proof of the main conjecture, it now suffices to compute matrix elements $\mathsf{a}^{\,p}_{-q}, \mathsf{d}^{-q}_{\;\;p}$. It turns out that a and d are represented in the Fourier basis by infinite Cauchy matrices. Finite determinants $Z^{[\pm]}_{\lambda,\mu,n}$ therefore have nice factorized expressions, finally leading to (1.4c), (1.4d).

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