Partition functions for reverse plane partitions derived from the two-dimensional Toda molecule

Shuhei KAMIOKA (Kyoto University)*

1. Introduction

Let λ be an (integer) partition or the corresponding Young diagram. A *reverse plane partition* of shape λ is a filling of cells in λ with nonnegative integers such that all rows and columns are weakly increasing. One of the most prominent results in the study of reverse (or ordinary) plane partitions is the discovery of *nice* generating functions, namely those which can be nicely factored.

The first discovery is due to MacMahon [6] who proved the triple product formula

$$\sum_{\pi} q^{|\pi|} = \prod_{i=1}^{r}\prod_{j=1}^{c}\prod_{k=1}^{n} \frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}} \tag{1}$$

for plane partitions π of $r \times c$ rectangular shape with parts at most n. MacMahon's study on plane partitions was later revived by Stanley [7]. Among his vast amounts of contributions a nice generating function involving the trace statistic is of great importance [8]. Stanley's trace generating function was much refined by Gansner [1] who derived the multi-trace generating function

$$\sum_{\pi \in \mathrm{RPP}(\lambda)} \prod_{\ell=1-r}^{c-1} x_\ell^{\mathsf{tr}_\ell(\pi)} = \prod_{(i,j)\in\lambda} \frac{1}{1-\prod_{\ell=j-\lambda'_j}^{\lambda_i-i} x_\ell} \tag{2}$$

where $\mathrm{RPP}(\lambda)$ denotes the set of reverse plane partitions $\pi = (\pi_{i,j})$ of shape λ with r rows and c columns, $\mathsf{tr}_\ell(\pi) = \sum_{-i+j=\ell} \pi_{i,j}$ the ℓ-trace, and λ' the shape conjugate with λ. Note that the parts of plane partitions considered in (1) are bounded from above by n but the parts of reverse plane partitions in (2) are unbounded. So the multi-trace generating function (2) is not a generalization of MacMahon's formula (1), and vice versa.

The aim of this study is to answer the question: Why (reverse) plane partitions admit so nice generating functions? Of course the question has been answered from different viewpoints by many authors who give various proofs to the nice formulas, with the help of representation theory, by calculating determinants and Pfaffians, and so on. (See, e.g., [4].) We here have a new viewpoint by an integrable system, the *discrete two-dimensional (2D) Toda molecule* [3].

In this talk we clarify a close connection of reverse plane partitions with the discrete two-dimensional (2D) Toda molecule. Especially we show that a nice partition function for reverse plane partitions can be derived from each non-vanishing solution to the discrete 2D Toda molecule (Theorem 6 in Section 4) where reverse plane partitions considered are those of arbitrary shape with bounded parts. As a concrete example we derive a partition function which generalizes both MacMahon's triple product formula (1) and Gansner's multi-trace generating function (2) from a specific solution (Theorem 7 in Section 5).

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*e-mail: `kamioka.shuhei.3w@kyoto-u.ac.jp`

The key idea comes from a combinatorial interpretation of the discrete 2D Toda molecule in terms of non-intersecting lattice paths (Section 3). Note that Viennot [9] takes a similar approach to count non-intersecting Dyck paths by using the quotient-difference (qd) algorithm for Padé approximation, also known as the discrete (one-dimensional) Toda molecule.

2. Solutions to the discrete 2D Toda molecule

We see a brief review on the integrable dynamical system discussed throughout the paper. The discrete two-dimensional (2D) Toda molecule is one of the most typical integrable dynamical systems that was introduced as a discrete analogue of the Toda lattice [3]. The evolution of the discrete 2D Toda molecule is described by the difference equations

$$a_n^{(s,t+1)} + b_n^{(s+1,t)} = a_n^{(s,t)} + b_{n+1}^{(s,t)}, \tag{3a}$$

$$a_n^{(s,t+1)} b_{n+1}^{(s+1,t)} = a_{n+1}^{(s,t)} b_{n+1}^{(s,t)}, \tag{3b}$$

$$(s,t) \in \mathbb{Z}^2, \quad n \in \mathbb{Z}_{\geq 0}, \quad b_0^{(s,t)} = 0. \tag{3c}$$

We introduce the tau function for (3) through the dependent variable transformation

$$a_n^{(s,t)} = \frac{\tau_{n+1}^{(s+1,t)} \tau_n^{(s,t)}}{\tau_n^{(s+1,t)} \tau_{n+1}^{(s,t)}}, \qquad b_n^{(s,t)} = \frac{\tau_{n-1}^{(s,t+1)} \tau_{n+1}^{(s,t)}}{\tau_n^{(s,t+1)} \tau_n^{(s,t)}} \tag{4}$$

with $\tau_0^{(s,t)} = 1$. We then obtained from (3) the bilinear form of the discrete 2D Toda molecule

$$\tau_{n-1}^{(s+1,t+1)} \tau_{n+1}^{(s,t)} - \tau_n^{(s+1,t+1)} \tau_n^{(s,t)} + \tau_n^{(s+1,t)} \tau_n^{(s,t+1)} = 0, \tag{5a}$$

$$(s,t) \in \mathbb{Z}^2, \quad n \in \mathbb{Z}_{\geq 1}, \quad \tau_0^{(s,t)} = 1. \tag{5b}$$

The bilinear form (5) is solved by the determinant

$$\tau_n^{(s,t)} = \det_{0 \leq i,j < n} (f_{s+i,t+j}) = \begin{vmatrix} f_{s,t} & \cdots & f_{s,t+j} & \cdots & f_{s,t+n-1} \\ \vdots & & \vdots & & \vdots \\ f_{s+i,t} & \cdots & f_{s+i,t+j} & \cdots & f_{s+i,t+n-1} \\ \vdots & & \vdots & & \vdots \\ f_{s+n-1,t} & \cdots & f_{s+n-1,t+j} & \cdots & f_{s+n-1,t+n-1} \end{vmatrix} \tag{6}$$

where $f = f_{i,j}$ is an arbitrary function defined over $\mathbb{Z}^2$. The discrete 2D Toda molecule (3) is therefore solved by $a_n^{(s,t)}$, $b_n^{(s,t)}$ given by (4) with (6) provided that the determinant never vanishes. Conversely there exists a function f on $\mathbb{Z}^2$ which satisfies (4) with (6) for any solution $a_n^{(s,t)} \neq 0$, $b_n^{(s,t)} \neq 0$ to (3). It is not difficult to see the following correspondence between $a_n^{(s,t)}$, $b_n^{(s,t)}$ and f.

Proposition 1. *For each solution $a_n^{(s,t)} \neq 0$, $b_n^{(s,t)} \neq 0$ to the discrete 2D Toda molecule* (3) *there exists a function $f = f_{i,j}$ on $\mathbb{Z}^2$ which gives the same solution through* (4) *with* (6)*. Moreover such an f is uniquely determined up to the transformation $f_{i,j} \to \varphi_j f_{i,j}$ by any non-vanishing function $\varphi = \varphi_j$ on $\mathbb{Z}$.*

Giving a non-vanishing solution $a_n^{(s,t)}$, $b_n^{(s,t)}$ to the discrete 2D Toda molecule is thus essentially equivalent to giving a function f over $\mathbb{Z}^2$.

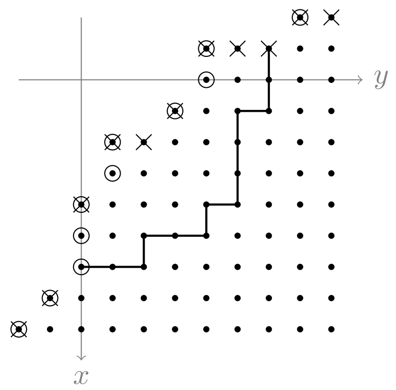

Figure 1: A regular subset $\mathbb{L}$ of $\mathbb{Z}^2$ and a lattice path. North and west boundary points are marked by circles and crosses respectively. Convex corners are those doubly marked.

3. Lattice path combinatorics

We adopt a matrix-like coordinate to draw a square lattice $\mathbb{Z}^2$ where the nearest neighbors $(i+1,j)$, $(i-1,j)$, $(i,j+1)$ and $(i,j-1)$ of a lattice point (i,j) are located on the south, north, east and west of (i,j) respectively. We call a subset $\mathbb{L}$ of $\mathbb{Z}^2$ *regular* such that (i) if $(i,j) \in \mathbb{L}$ then $(i+k,j+k) \in \mathbb{L}$ for all $k \geq 1$; (ii) if $(i,j) \in \mathbb{L}$ then $(i-k,j) \notin \mathbb{L}$ and $(i,j-k) \notin \mathbb{L}$ for some $k \geq 1$. We call a point $(i,j) \in \mathbb{L}$ a *north boundary point* if $(i-1,j) \notin \mathbb{L}$; similarly a *west boundary point* if $(i,j-1) \notin \mathbb{L}$. We call a point $(i,j) \in \mathbb{L}$ a *convex corner* if (i,j) is a north and south boundary point. The interest is in lattice paths on a regular subset $\mathbb{L}$ of $\mathbb{Z}^2$ consisting of north and east steps. See Figure 1 for example.

We think of a regular subset $\mathbb{L}$ of $\mathbb{Z}^2$ as a graph with vertices $\mathbb{L}$ and edges connecting nearest neighbors. We determine the weights of edges by using a solution $a_n^{(s,t)}$, $b_n^{(s,t)}$ to the discrete 2D Toda molecule (3) as follows.

(a) The vertical edge with north endpoint at (i,j) is weighted by $a_n^{(i-n,j-n)}$ if $(i-n,j-n)$ is a west boundary point of $\mathbb{L}$.

(b) The vertical edge with south endpoint at (i,j) is weighted by $b_n^{(i-n,j-n)}$ if $(i-n,j-n)$ is a north boundary point of $\mathbb{L}$.

(c) Every horizontal edge is weighted by 1.

See Figure 2 for example. We define the weight $w(\mathbb{L};a,b;P)$ of a lattice path P on $\mathbb{L}$ to be the product of the weights of all edges passed by P. We conventionally consider empty paths P with no steps for which $w(\mathbb{L};a,b;P) = 1$. For $(i,j) \in \mathbb{L}$ and $(k,\ell) \in \mathbb{L}$ we further define

$$g(\mathbb{L};a,b;i,j;k,\ell) = \sum_P w(\mathbb{L};a,b;P) \tag{7}$$

where the sum ranges over all lattice paths on $\mathbb{L}$ going from (i,j) to (k,ℓ).

Let $\mathsf{x}(j)$ denote the x-coordinate (or the vertical —) of the north boundary point of $\mathbb{L}$ with y-coordinate (or horizontal —) equal to j; let $\mathsf{y}(i)$ the y-coordinate of the west boundary point of $\mathbb{L}$ with x-coordinate equal to i. The following theorem gives a combinatorial interpretation of the discrete 2D Toda molecule and refines Proposition 1.

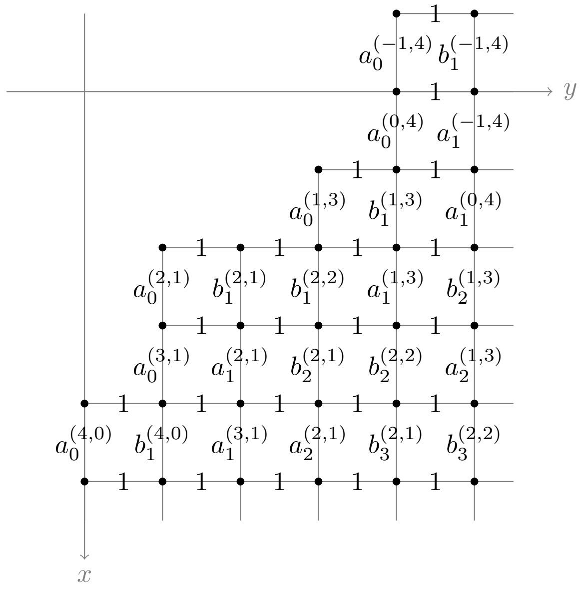

Figure 2: The weights of edges.

Theorem 2. *Let $a_n^{(s,t)} \neq 0$, $b_n^{(s,t)} \neq 0$ be a solution to the discrete 2D Toda molecule* (3)*, and let a function $f = f_{i,j}$ on $\mathbb{Z}^2$ give the same solution through* (4) *with* (6)*. Let $\mathbb{L}$ be a regular subset of $\mathbb{Z}^2$. For $(i,j) \in \mathbb{L}$ then*

$$\frac{f_{i,j}}{f_{\mathsf{x}(j),j}} = g(\mathbb{L}; a, b; i, \mathsf{y}(i); \mathsf{x}(j), j). \tag{8}$$

In order to prove the theorem we use the following lemma.

Lemma 3. *Let $\mathbb{L}$ be a regular subset of $\mathbb{Z}^2$ with a convex corner $(s,t) \in \mathbb{L}$. Let $\mathbb{L}'$ denote the regular subset of $\mathbb{Z}^2$ obtained from $\mathbb{L}$ by deleting (s,t). For (i,j) and (k,ℓ) in $\mathbb{L}'$ with $i-j \neq s-t$ and $k-\ell \neq s-t$ then*

$$g(\mathbb{L}; a, b; i, j; k, \ell) = g(\mathbb{L}'; a, b; i, j; k, \ell). \tag{9}$$

Proof. The difference between $\mathbb{L}$ and $\mathbb{L}'$ is only in the existence and the absence of the convex corner (s,t), and the weights of vertical edges between the two diagonal lines $d_- : y - x = t - s - 1$ and $d_+ : y - x = t - s + 1$. (The vertical edges between d_- and d_+ are weighted by $a_n^{(s,t)}$, $b_n^{(s,t)}$ on $\mathbb{L}$ and by $a_n^{(s,t+1)}$, $b_n^{(s+1,t)}$ on $\mathbb{L}'$.) Assume that $i - j \neq s - t$ and $k - \ell \neq s - t$ meaning that (i,j) and (k,ℓ) is outside the region between d_- and d_+. (Those may be on $d_\pm$.) If both (i,j) and (k,ℓ) are either in the south of d_- or in the north of d_+ then the identity (9) clearly holds since lattice paths going from (i,j) to (k,ℓ) never enter the region between d_- and d_+. In the rest of the proof we thus assume that (i,j) is in the south of d_- and (k,ℓ) in the north of d_+.

Each lattice path P going from (i,j) to (k,ℓ) is uniquely divided into three subpaths: P_- from (i,j) to d_-, Q of two steps between d_- and d_+ and P_+ from d_+ to (k,ℓ). Obviously $w(P_\pm) = w'(P_\pm)$ where w and w' are abbreviations of $w(\mathbb{L}; a, b; \cdot)$ and $w(\mathbb{L}'; a, b; \cdot)$ respectively. The proof of (9) thus amounts to showing that $g(i,j;k,\ell) = g'(i,j;k,\ell)$ for each (i,j) on d_- and (k,ℓ) on d_- where g and g' are abbreviations of $g(\mathbb{L}; a, b; \cdot)$ and $g(\mathbb{L}'; a, b; \cdot)$ respectively. Since Q is of two steps we have only three cases: (i) $(i,j) = (s+n, t+n-1)$ and $(k,\ell) = (s+n, t+n+1)$ for some $n \geq 1$; (ii)

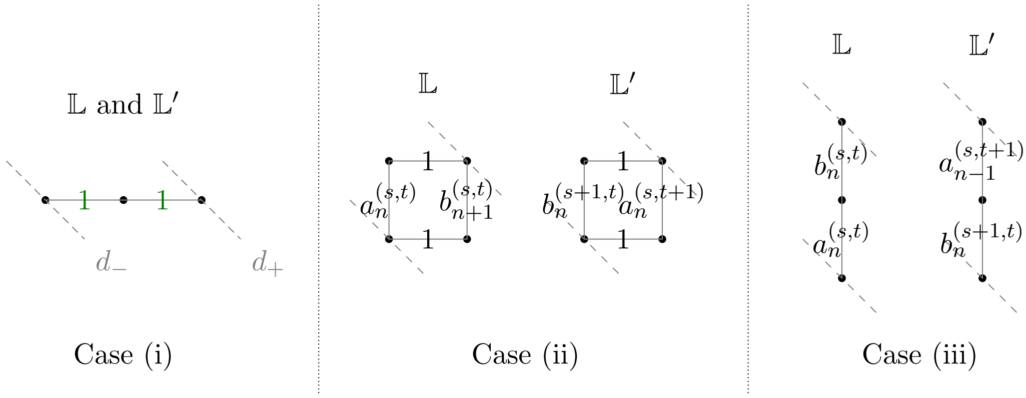

Figure 3: Proof of Lemma 3.

$(i,j) = (s+n+1, t+n)$ and $(k,\ell) = (s+n, t+n+1)$ for some $n \geq 0$; (iii) $(i,j) = (s+n+1, t+n)$ and $(k,\ell) = (s+n-1, t+n)$ for some $n \geq 1$. See Figure 3.

Case (i): The unique lattice path going from $(i,j) = (s+n, t+n-1)$ to $(k,\ell) = (s+n, t+n+1)$ of two east steps is both on $\mathbb{L}$ and on $\mathbb{L}'$. Thus $g(i,j;k,\ell) = g'(i,j;k,\ell) = 1$.

Case (ii): There are two lattice paths going from $(i,j) = (s+n+1, t+n)$ to $(k,\ell) = (s+n, t+n+1)$ one of which is Q_1 going north and east, the other is Q_2 going east and north. If $n \geq 1$ then Q_1 and Q_2 are both on $\mathbb{L}$ and on $\mathbb{L}'$, and $w(Q_1) = a_n^{(s,t)}$, $w(Q_2) = b_{n+1}^{(s,t)}$, $w'(Q_1) = a_n^{(s,t+1)}$ and $w'(Q_2) = b_n^{(s+1,t)}$. Thus $g(i,j;k,\ell) = a_n^{(s,t)} + b_{n+1}^{(s,t)} = a_n^{(s,t+1)} + b_n^{(s+1,t)} = g'(i,j;k,\ell)$ because of (3b). If $n = 0$ then Q_1 and Q_2 are on $\mathbb{L}$ while only Q_2 on $\mathbb{L}'$, and $w(Q_1) = a_0^{(s,t)}$, $w(Q_2) = b_1^{(s,t)}$ and $w'(Q_1) = a_0^{(s,t+1)}$. (Q_1 is not on $\mathbb{L}'$ since Q_1 passes through $(s,t) \notin \mathbb{L}'$.) Thus $g(i,j;k,\ell) = a_0^{(s,t)} + b_1^{(s,t)} = a_0^{(s,t+1)} = g'(i,j;k,\ell)$ because of (3a) with (3c).

Case (iii): The unique lattice path going from $(i,j) = (s+n+1, t+n)$ to $(k,\ell) = (s+n-1, t+n)$ of two north steps is both on $\mathbb{L}$ and on $\mathbb{L}'$. The weight of the lattice path is $a_n^{(s,t)} b_n^{(s,t)}$ on $\mathbb{L}$ and $a_{n-1}^{(s,t+1)} b_n^{(s+1,t)}$ on $\mathbb{L}'$. Thus $g(i,j;k,\ell) = a_n^{(s,t)} b_n^{(s,t)} = a_{n-1}^{(s,t+1)} b_n^{(s+1,t)} = g'(i,j;k,\ell)$ because of (3b). $\square$

Proof of Theorem 2. Let $\mathbb{L}'$ denote the regular subset of $\mathbb{Z}^2$ defined by $\mathbb{L}' = \mathbb{L} \setminus \{(s,t) \in \mathbb{L};\ s < i \text{ and } t < j\}$. From Lemma 3 then $g(\mathbb{L}; a, b; i, \mathsf{y}(i); \mathsf{x}(j), j) = g(\mathbb{L}'; a, b; i, \mathsf{y}(i); \mathsf{x}(j), j)$ because $\mathbb{L}'$ can be obtained from $\mathbb{L}$ by iterative deletion of convex corners. A lattice path going from $(i, \mathsf{y}(i))$ to $(\mathsf{x}(j), j)$ on $\mathbb{L}'$ is unique because such a lattice path cannot turn north until (i,j) and cannot turn east from (i,j). The weight of the unique lattice path on $\mathbb{L}'$ implies that $g(\mathbb{L}'; a, b; i, \mathsf{y}(i); \mathsf{x}(j), j) = \prod_{k=\mathsf{x}(j)}^{i-1} a_0^{(k,j)}$. The last product is equal to $f_{i,j}/f_{\mathsf{x}(j),j}$ because $a_0^{(k,j)} = f_{k+1,j}/f_{k,j}$ from (4) and (6). $\square$

Theorem 2 admits a combinatorial interpretation of the determinant $\tau_n^{(s,t)}$ by means of Gessel–Viennot–Lindström's method [2, 5]. For $(s,t) \in \mathbb{L}$ and $n \geq 0$ we define $\mathrm{LP}(\mathbb{L}, s, t, n)$ to be the set of n-tuples $(P_0, \ldots, P_{n-1})$ of lattice paths on $\mathbb{L}$ such that (i) P_k goes from $(s+k, \mathsf{y}(s)+k)$ to $(\mathsf{x}(t)+k, t+k)$ for each $0 \leq k < n$, and (ii) $P_0, \ldots, P_{n-1}$ are *non-intersecting*: $P_j \cap P_k = \emptyset$ if $j \neq k$. See Figure 4, the second figure shows an n-tuple of non-intersecting lattice paths.

Theorem 4. *Let $a_n^{(s,t)} \neq 0$, $b_n^{(s,t)} \neq 0$ be a solution to the discrete 2D Toda molecule* (3)*, and let a function $f = f_{i,j}$ on $\mathbb{Z}^2$ give the same solution through* (4) *with* (6)*. Let*

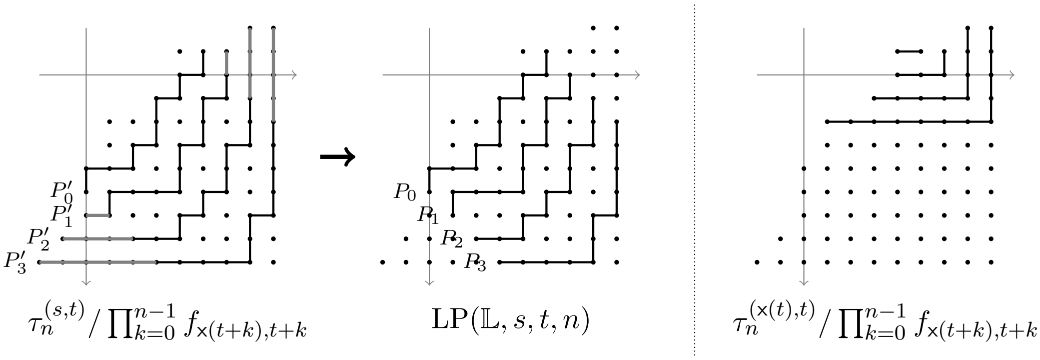

Figure 4: Proof of Theorem 4 where $(s,t) = (5,4)$ and $n = 4$. In the first figure the steps in gray are frozen due to the non-intersecting condition.

$\mathbb{L}$ *be a regular subset of* $\mathbb{Z}^2$. *For* $(s,t) \in \mathbb{L}$ *and* $n \geq 0$ *then*

$$\frac{\tau_n^{(s,t)}}{\tau_n^{(\mathsf{x}(t),t)}} = \sum_{(P_0,\dots,P_{n-1}) \in \mathrm{LP}(\mathbb{L},s,t,n)} \prod_{k=0}^{n-1} w(\mathbb{L}; a, b; P_k) \tag{10}$$

where $\tau_n^{(s,t)} = \det_{0 \leq i,j < n}(f_{s+i,t+j})$.

Proof. Gessel–Viennot–Lindström's method yields from Theorem 2 that

$$\frac{\tau_n^{(s,t)}}{\prod_{k=0}^{n-1} f_{\mathsf{x}(t+k),t+k}} = \sum_{(P_0',\dots,P_{n-1}')} \prod_{k=0}^{n-1} w(\mathbb{L}; a, b; P_k') \tag{11}$$

where the sum ranges over all n-tuples $(P_0', \dots, P_{n-1}')$ of non-intersecting lattice paths on $\mathbb{L}$ such that P_k' goes from $(s+k, \mathsf{y}(s+k))$ to $(\mathsf{x}(t+k), t+k)$ for each $0 \leq k < n$. Eliminating the steps frozen due to the non-intersecting condition we obtain $(P_0, \dots, P_{n-1}) \in \mathrm{LP}(\mathbb{L}, s, t, n)$, see Figure 4 for example. The weight of the eliminated frozen steps is equal to the weight of a unique configuration of non-intersecting lattice paths from $\tau_n^{(\mathsf{x}(t),t)} / \prod_{k=0}^{n-1} f_{\mathsf{x}(t+k),t+k}$, see the first and the last figures in Figure 4 for example. Thus $\tau_n^{(s,t)}$ is equal to the right-hand side of (10) multiplied by $\tau_n^{(\mathsf{x}(t),t)}$. □

Note that the left-hand side of (10) can be expressed as

$$\frac{\tau_n^{(s,t)}}{\tau_n^{(\mathsf{x}(t),t)}} = \prod_{i=1}^{s-\mathsf{x}(t)} \prod_{k=1}^{n} a_{k-1}^{(s-i,t)} \tag{12}$$

from (4). We can readily evaluate the sum in (10), a partition function for non-intersecting lattice paths, by using this formula.

4. Multiplicative partition functions for reverse plane partitions

Let λ be a partition and let $n \geq 0$. We write $\mathrm{RPP}(\lambda, n)$ for the set of reverse plane partitions of shape λ with parts at most n. Let r and c denote the numbers of rows and columns in λ respectively. We then define a regular subset $\mathbb{L}(\lambda)$ of $\mathbb{Z}^2$ by

$$\mathbb{L}(\lambda) = \{(i,j) \in \mathbb{Z}_{\geq 0}^2;\ j \geq c - \lambda_{r-i}\} \tag{13}$$

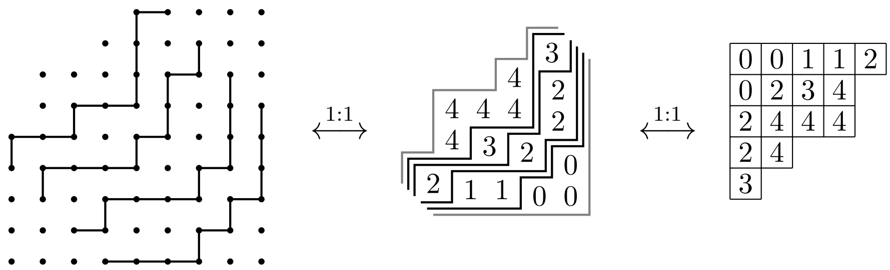

Figure 5: The bijection between $\mathrm{LP}(\mathbb{L}(\lambda), r, c, n)$ and $\mathrm{RPP}(\lambda, n)$ where $\lambda = (5, 4, 4, 2, 1)$ with $r = 5$ rows and $c = 5$ columns, and $n = 4$.

where λ_i denotes the i-th part of λ for $1 \le i \le r$ and $\lambda_i = c$ for $i \le 0$. There is a bijection between $\mathrm{LP}(\mathbb{L}(\lambda), r, c, n)$ and $\mathrm{RPP}(\lambda, n)$ which is described as follows. Given an n-tuple $(P_0, \dots, P_{n-1}) \in \mathrm{LP}(\mathbb{L}(\lambda), r, c, n)$ of non-intersecting lattice paths on $\mathbb{L}(\lambda)$

(i) move the lattice path P_k northwest by $(-k, -k)$ for each $0 \le k < n$;

(ii) fill in the cells between P_{n-k-1} and P_{n-k} with k for each $0 \le k \le n$ where P_{-1} is the lattice path going from $(r, 0)$ to $(0, c)$ along the border of $\mathbb{L}(\lambda)$ and P_n is that going east from $(r, 0)$, turning north at (r, c) and going north to $(0, c)$;

(iii) rotate 180° to obtain a reverse plane partition in $\mathrm{RPP}(\lambda, n)$.

Figure 5 demonstrates the bijection by an example. It should be noted that this bijection is essentially the same as the classical interpretation of plane partitions by "zig-zag" non-intersecting paths [4].

We set up weight for reverse plane partitions which is equivalent to the weight for lattice paths defined in Section 3. Let $\lambda' = (\lambda'_1, \dots, \lambda'_c)$ denote the partition conjugate with λ. We define $\alpha_{i,j}$ by

$$\alpha_{i+k,\lambda_i+k} = a_{n-k-1}^{(r-i,c-\lambda_i)}, \qquad \alpha_{\lambda'_j+k,j+k-1} = b_{n-k}^{(r-\lambda'_j,c-j)} \tag{14}$$

for $1 \le i \le r$, $1 \le j \le c$ and $k < n$ where $a_n^{(s,t)} \ne 0$, $b_n^{(s,t)} \ne 0$ is a solution to the discrete 2D Toda molecule (3). We then define the weight of a reverse plane partition π by

$$v(\lambda, n; a, b; \pi) = \prod_{(i,j)\in\lambda} v_{i,j}(\lambda, n; a, b; \pi) \qquad \text{with} \tag{15a}$$

$$v_{i,j}(\lambda, n; a, b; \pi) = \prod_{k=1}^{\pi_{i,j}} \frac{\alpha_{i+k-1,j+k-2}}{\alpha_{i+k-1,j+k-1}}. \tag{15b}$$

Lemma 5. *Let λ be a partition with r rows and c columns, and let $n \ge 0$. Assume that $\pi \in \mathrm{RPP}(\lambda, n)$ and $(P_0, \dots, P_{n-1}) \in \mathrm{LP}(\mathbb{L}(\lambda), r, c, n)$ corresponds to each other by the bijection. Then*

$$v(\lambda, n; a, b; \pi) = \frac{\prod_{k=0}^{n-1} w(\mathbb{L}(\lambda); a, b; P_k)}{\prod_{i=1}^{r} \prod_{k=1}^{n} a_{k-1}^{(r-i,c-\lambda_i)}}. \tag{16}$$

Sketch of proof. Actually $\alpha_{i,j}$ is defined so that $v(\pi) = v(\lambda, n; a, b; \pi)$ is proportional to $\prod_{k=0}^{n-1} w(P_k)$ with $w(P) = w(\mathbb{L}(\lambda); a, b; P)$. That is, there exists a constant κ such that $v(\pi) = \kappa \prod_{k=0}^{n-1} w(P_k)$. From (15), $v(\lambda, n; a, b; \pi^{\emptyset}) = 1$ for the *empty* reverse plane partition $\pi^{\emptyset} \in \mathrm{RPP}(\lambda, n)$ whose parts are all 0. Thus $\kappa^{-1} = \prod_{k=0}^{n-1} w(P_k^{\emptyset})$ where $(P_0^{\emptyset}, \dots, P_{n-1}^{\emptyset}) \in \mathrm{LP}(\mathbb{L}(\lambda), r, c, n)$ corresponds to $\pi^{\emptyset}$ by the bijection. We observe that $P_0^{\emptyset}$ goes from $(r, 0)$ to $(0, c)$ along the border of $\mathbb{L}(\lambda)$, and $P_1^{\emptyset}, \dots, P_{n-1}^{\emptyset}$ are copies of $P_0^{\emptyset}$. Especially $w(P_k^{\emptyset}) = \prod_{i=1}^{r} a_k^{(r-i, c-\lambda_i)}$ and hence κ^{-1} is equal to the denominator of the right-hand side of (16). $\square$

The following is the main theorem of this paper.

Theorem 6. *Let $a_n^{(s,t)} \neq 0$, $b_n^{(s,t)} \neq 0$ be a solution to the discrete 2D Toda molecule (3). Let λ be a partition with r rows and c columns, and let $n \geq 0$. Then*

$$\sum_{\pi \in \mathrm{RPP}(\lambda, n)} v(\lambda, n; a, b; \pi) = \prod_{i=1}^{r} \prod_{k=1}^{n} \frac{a_{k-1}^{(r-i, c)}}{a_{k-1}^{(r-i, c-\lambda_i)}}. \tag{17}$$

Proof. This theorem is a translation of Theorem 4 via the bijection with the help of (12) and Lemma 5. $\square$

Theorem 6 allows us to find a multiplicative partition function for reverse plane partitions of arbitrary shape with bounded parts from each non-vanishing solution to the discrete 2D Toda molecule (3).

5. An example

The discrete 2D Toda molecule (3) has the solution

$$a_n^{(s,t)} = [p]_{s+1}^{s+n} (1 - a [p]_1^s [q]_1^{t+n}), \tag{18a}$$
$$b_n^{(s,t)} = a [p]_1^{s+n-1} [q]_1^t (1 - [q]_{t+1}^{t+n}) \tag{18b}$$

with the notation that $[z]_m^n = \prod_{\ell=m}^{n} z_\ell$ if $m \leq n$, $[z]_m^n = 1$ if $m = n+1$ and $[z]_m^n = \prod_{\ell=n}^{m-1} z_\ell^{-1}$ if $m \geq n+2$. The solution involves indeterminates a and p_ℓ, q_ℓ for $\ell \in \mathbb{Z}$ as parameters.

Let λ be a partition with r rows and c columns. Assume that

$$a = [x]_{c-\lambda'_c}^{\lambda_r - r}, \qquad p_i = [x]_{c-r+i-\mu_i}^{c-r+i-\mu_{i+1}}, \qquad q_j = [x]_{\mu'_{j+1}-j-r+c}^{\mu'_j - j - r + c}. \tag{19}$$

We then have a solution

$$a_n^{(s,t)} = [x]_{c-r+s+1-\mu_{s+1}}^{c-r+s+n-\mu_{s+n+1}} (1 - [x]_{\mu'_{t+n+1}-t-n-r+c}^{c-r+s-\mu_{s+1}}), \tag{20a}$$
$$b_n^{(s,t)} = [x]_{\mu'_{t+1}-t-r+c}^{c-r+s+n-1-\mu_{s+n}} (1 - [x]_{\mu'_{t+n+1}-t-n-r+c}^{\mu'_{t+1}-t-1-r+c}). \tag{20b}$$

Let $n \geq 0$. The solution (20) yields the weight of (15) with (14) given by

$$v(\lambda, n; a, b; \pi) = \prod_{\ell=1-r}^{c-1} x_\ell^{\mathrm{tr}_\ell(\pi)} \prod_{(i,j) \in \lambda} \prod_{k=1}^{\pi_{i,j}} \frac{1 - [x]_{-n+j+k-1-\lambda'_{-n+j+k-1}}^{j-i-1}}{1 - [x]_{-n+j+k+\lambda'_{-n+j+k}}^{j-i}}. \tag{21}$$

As an instance of Theorem 6 we obtain the following multiplicative partition function for reverse plane partition.

Theorem 7. *Let λ be a partition with r rows and c columns, and let $n \geq 0$. Then*

$$\sum_{\pi \in \mathrm{RPP}(\lambda,n)} v(\lambda, n; a, b; \pi) = \prod_{(i,j)\in\lambda} \frac{1 - [x]^{\lambda_i - i}_{-n+j-\lambda'_{-n+j}}}{1 - [x]^{\lambda_i - i}_{j-\lambda'_j}} \tag{22}$$

where the weight $v(\lambda, n; a, b; \pi)$ is given by (21).

Proof. Substituting the solution (20) for the right-hand side of (17) we get

$$\prod_{i=1}^{r}\prod_{k=1}^{n} \frac{1 - [x]^{\lambda_i - i}_{-k+1-\lambda'_{-k+1}}}{1 - [x]^{\lambda_i - i}_{-k+1+\lambda_i-\lambda'_{-k+1+\lambda_i}}} = \prod_{i=1}^{r}\prod_{j=1}^{\lambda_i}\prod_{k=1}^{n} \frac{1 - [x]^{\lambda_i - i}_{j-k-\lambda'_{j-k}}}{1 - [x]^{\lambda_i - i}_{j-k+1-\lambda'_{j-k+1}}} \tag{23a}$$

$$= \prod_{i=1}^{r}\prod_{j=1}^{\lambda_i} \frac{1 - [x]^{\lambda_i - i}_{-n+j-\lambda'_{-n+j}}}{1 - [x]^{\lambda_i - i}_{j-\lambda'_j}}. \tag{23b}$$

The last product is the same as the right-hand side of (22). □

The multiplicative partition function in Theorem 7 generalizes the multi-trace generating function (2) by Gansner. Indeed (22) reduces into (2) as $n \to \infty$ because $\lim_{n\to\infty}[x]^{\text{const.}}_{-n+\text{const.}} = 1$ as formal power series, $\lim_{n\to\infty}\lambda'_{-n} = r$ and $\lim_{n\to\infty} v(\lambda, n; a, b; \pi) = \prod_{\ell=1-r}^{c-1} x_\ell^{\mathrm{tr}_\ell(\pi)}$.

Assuming $x_\ell = q$ for $\ell \in \mathbb{Z}$ we obtain the partition function

$$\sum_{\pi \in \mathrm{RPP}(\lambda,n)} v(\lambda, n; a, b; \pi) = \prod_{(i,j)\in\lambda} \frac{1 - q^{\lambda_i + \lambda'_{j-n} - i - j + n + 1}}{1 - q^{\lambda_i + \lambda'_j - i - j + 1}} \quad \text{with} \tag{24a}$$

$$v(\lambda, n; a, b; \pi) = q^{|\pi|} \prod_{(i,j)\in\lambda}\prod_{k=1}^{\pi_{i,j}} \frac{1 - q^{n-i-k+1+\lambda'_{-n+j+k-1}}}{1 - q^{n-i-k+1+\lambda'_{-n+j+k}}} \tag{24b}$$

from (21) and (22). If $\lambda = (c^r)$, an $r \times c$ rectangular shape, that becomes the triple product formula (1) by MacMahon. The partition function (24) is thus regarded as a generalization of (1) for reverse plane partitions of arbitrary shape.

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